K-theory and C^*-algebras

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Introduction

The study of the algebra of bounded operators on a Hilbert space is an important area of functional analysis. In order to study the algebraic and topological structures of this kind of algebras, one wants to define some more general algebras : C^*-algebras. They basically are Banach algebras equipped with an involution, which generalize the notion of adjunction for operators on a Hilbert space. These algebras, and more generally operator algebras, are an important piece of non-commutative geometry and its application to Physics for instance. In order to study and discriminate the topological properties of C^*-algebras, one would like to have a tool, a topological invariant, as we have the fundamental group for topological spaces with which we can say wether two topological spaces are not homotopic (and so not homeomorphic). In operator algebras, we have K-theory. It allows us to classify some C^*-algebras. As we will see, as a functor from the category of C^*-algebras to the category of abelian groups, K-theory has numerous interesting properties such as homotopy invariance, stability, half and split exactness. Besides, there are powerful tools to compute the K-groups of a C^*-algebras, like the 6-term exact sequence. K-theory was first developed by Alexander Grothendieck for algebraic geometry in 1957. Then, in 1961, Micheal Atiyah and Friedrich Hirzebruch applied the same construction to vector bundles : this is topological K-theory. K-theory for C^*-algebras, and more generally for operator algebras (it can be constructed with Banach algebras) is a generalization of topological K-theory and therefore computations in these two cases are related, by the Gelfand-Naimark theorem for instance, which basically maps a commutative C^*-algebra to a topological space. This report is an introduction to K-theory for C^*-algebras.

In the first section, we will define C^*-algebras and state some basic properties and important theorems such as the Gelfand-Naimark theorem. Projections, which are the elements of C^*-algebras we will need to define K-theory, will be studied. Then we will define tensor products of C^*-algebras and study their behavior. In the second section, we will define some preliminary tools like categories, direct limits of topological spaces and the Grothendieck construction. Then we will be able to define the first K-group K_0 and study its functoriality, and define higher K-functors, which shares many properties of K_0, as wee will see. Finally, in the third section, we will study the half-infinite exactness of K-theory and the Toeplitz algebra in order to prove one of the most powerful tools to compute K-theory of some C^*-algebras : the Bott periodicity theorem and then the 6-term exact sequence as a consequence.

1 C^*-algebras

1.1 Basic definitions and main statements

We introduce here some of the basic definitions and results about C^*-algebras, which one can find in many books, like [Mur90], [Bla05] and [Dix69]. More of the basic theory, and spectral theory, of C^*-algebras and, more generally, Banach algebras can be found in the first two books.

Definition 1.1.1. A C^*-algebra A is an algebra over $\mathbb{C}$ with a norm and an involution $*$ such that A is complete and such that $\|ab\| \leq \|a\| \|b\|$ and $\|a^*a\| = \|a\|^2$ for every $a, b \in A$. A is called unital if A has a multiplicative identity.

Remark 1.1.1. A $*$-algebra is just an algebra over $\mathbb{C}$ with an involution.

Example 1.1.1.

- The unital algebra $C(X)$ of continuous functions defined on a compact Hausdorff space X, with complex conjugacy as an involution.
- The algebra $C_0(X)$ of continuous functions defined on a locally compact Hausdorff space X and vanishing at infinity.
- The matrix algebra $\mathcal{M}_n(\mathbb{C})$, with the operator norm and conjugate-transpose.
- $\mathcal{M}_n(A)$ for any C^*-algebra A, with the operator norm and involution-transpose.
- The algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H, endowed with the operator norm.

Remark 1.1.2. One can easily verify that the intersection of C^*-subalgebras of a C^*-algebra A is a C^*-algebra. Then we can define the C^*-subalgebra generated by a subset $S \subset A$ as the smallest C^*-subalgebra of A containing S.

The direct sum $A \oplus B$ of two C^*-algebras A and B, endowed with the entrywise involution and the norm

$$\|(a, b)\| = \max(\|a\|, \|b\|)$$

is a C^*-algebra. And, if I is a closed ideal of A (in the following every ideal will be two-sided and closed), I and the quotient A/I are C^*-algebras, with usual norms. See [Bla05] for more details.

In the case of a non-unital C^*-algebra A, we will sometimes need to add a unit. Define $A^+ = A \oplus \mathbb{C}$ as a vector space, and endow it with the entrywise involution, the multiplication

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

and the norm

$$\|(a, \lambda)\| = \sup\{\|ab + \lambda b\|, b \in A, \|b\| = 1\}$$

Then A^+ is a unital C^*-algebra, whose A is a closed ideal, by the inclusion $a \mapsto (a, 0)$. Note that, if $\varphi : A \to B$ is a $*$-homomorphism, we can extend it to a $*$-homomorphism $\varphi^+ : A^+ \to B^+$ by $(a, \lambda) \mapsto (\varphi(a), \lambda)$. Thus, unitalization is a functor from the category of C^*-algebras to the category of unital C^*-algebras.

Definition 1.1.2. Let A be a unital C^*-algebra and $a \in A$. We call the spectrum of a the set

$$\mathrm{sp}(a) = \left\{\lambda \in \mathbb{C} \;\middle|\; a - \lambda 1 \notin \mathrm{Inv}(A^+)\right\}$$

Definition 1.1.3. Given two C^*-algebras A and B. A $*$-homomorphism, $\varphi : A \to B$ is an algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$. It will often be just called homomorphism when the context is clear.

Remark 1.1.3. By considering the spectral radius of an element of a C^*-algebra, one can prove that any $*$-homomorphism between C^*-algebras is continuous. Furthermore the image of a homomorphism between C^*-algebras is closed (see [Bla05] for a proof).

We have a natural notion of homotopy between two homomorphisms.

Definition 1.1.4. Let $f, g : A \to B$ two homomorphisms between C^*-algebras A and B. Then f, g are called homotopic if there is a homomorphism, called homotopy, $F : A \to C([0\,, 1], B)$ such that $ev_0 \circ F = f$ and $ev_1 \circ F = g$. Here ev_x denotes the evaluation map at x.

Hence we get a notion of homotopy equivalence between C^*-algebras.

Definition 1.1.5. Two C^*-algebras A and B are homotopic if there exist two homomorphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g$ is homotopic to id_B and $g \circ f$ is homotopic to id_B. We denote it by $A \approx B$ and we call f, g homotopy equivalences.

Definition 1.1.6. An element a of a C^*-algebra A is called :

- normal if $aa^* = a^*a$
- positive if a is normal and $\mathrm{sp}(a) \subset \mathbb{R}_+$
- unitary if $aa^* = a^*a = 1$ (in the case A unital)
- a projection if $a^2 = a = a^*$
- a partial isometry if v^*v is a projection

We denote by $\mathcal{U}(A)$ the set of unitary elements of A and by $\mathcal{P}(A)$ the set of projections, and by $\mathcal{U}_n(A)$ and $\mathcal{P}_n(A)$ the set of unitaries and the set of projections in $\mathcal{M}_n(A)$.
The following theorem is one of the most fundamental results of the theory of commutative C^*-algebras. It is useful to calculate the K-theory of some C^*-algebras, by making a link (which we will not see here) between K-theory of vector bundles and K-theory of C^*-algebras. One can find a proof in [Mur90].

Theorem 1.1.1 (Gelfand)**.** Every commutative C^*-algebra A is isometrically $*$-isomorphic to the C^*-algebra $C_0(X)$ for some locally compact Hausdorff space X.

Representation of C^*-algebras and, the Gelfand-Naimark theorem, are used all the time in the theory of C^*-algebras. It will allow us to see any C^*-algebra as a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

Definition 1.1.7. A representation of a C^*-algebra A is a pair (H, φ) where H is an Hilbert space and $\varphi : A \to \mathcal{B}(H)$ is a $*$-homomorphism. The representation (H, φ) is called faithful if φ is injective.

Remark 1.1.4. If $(H_\lambda, \rho_\lambda)_{\lambda \in \Lambda}$ is a family of representation, then their direct sum is a representation.

A proof of the following theorem can be found in [Mur90].

Theorem 1.1.2 (Gelfand-Naimark)**.** Every C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. If A is separable, H can be chosen to be separable.

Continuous functional calculus is another powerful tool for C^*-algebras. We will not prove the following proposition. However one can read more about continuous functional calculus in [Dix69].

Proposition 1.1.1. Let A be a unital C^*-algebra and $x \in A$ a normal element. Then we have the following homomorphism of C^*-algebras

$$\begin{array}{rcl} C(\mathrm{sp}(x)) & \longrightarrow & A \\ f & \longmapsto & f(x) \end{array}$$

and $\forall f \in C(\mathrm{sp}(x)),\ \mathrm{sp}(f(x)) = f(\mathrm{sp}(x))$.

Finally, let us state the polar decomposition in a unital C^*-algebra, which we will need in the following. Just mention the existence of a unique square root of a positive element, whose a proof can be found in [Mur90].

Proposition 1.1.2. Let A be a C^*-algebra and $a \in A$ positive. Then there exists a unique element $b \in A$ such that $b^2 = a$. This element is called the square root of a, denoted by $a^{1/2}$.

Given an element a of a C^*-algebra A, we call the absolute value of a the element $|a| = (a^*a)^{1/2}$. From this comes the polar decomposition. One can read a proof in [RLL00].

Proposition 1.1.3. Let A be a unital C^*-algebra and $a \in A$ invertible. Then $|a|$ is invertible and $u = a\,|a|^{-1}$ is unitary. Note that $a = u\,|a|$. Moreover, the defined map $u : \mathrm{GL}(A) \to \mathcal{U}(A)$ is continuous.

1.2 Projections

The group K_0 of a C^*-algebra is defined from equivalent classes of projections. In this subsection, which is based on [NdK16], we will introduce them and see some their important properties, which will give us a better understanding of K_0 and several ways of representing the elements of K_0. In this subsection A will be a unital C^*-algebra. We define three equivalence relations on $\mathcal{P}(A)$:

- $p \sim_h q$ if there is a path in $\mathcal{P}(A)$ connecting p and q ; we say that p and q are path connected,
- $p \sim_u q$ if there exists a unitary $u \in \mathcal{U}(A)$ such that $p = uqu^*$; we say that p and q are unitarily equivalent,
- $p \sim q$ if there exists $x \in A$ such that $p = x^*x$ and $q = xx^*$; we say that p and q are Murray-von Neumann equivalent.

Note that, if two projections are equivalent in any of these ways in $\mathcal{P}_n(A)$, they are also equivalent in $\mathcal{P}_{n+1}(A)$, under the embedding $a = \mathrm{diag}(a, 0)$. In fact, any matrix $a \in \mathcal{M}_n(A)$ can be seen as $\mathrm{diag}(a, 0)$ (which we will sometimes still denote by a, without ambibuity) in $\mathcal{M}_m(A)$, for $m \geq n$. For any two matrices $p \in \mathcal{M}_n(A)$ and $q \in \mathcal{M}_m(A)$, define

$$p \oplus q = \mathrm{diag}(p, q) = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

This operation is associative and, if p, q are projections, so is $p \oplus q$.

Proposition 1.2.1. Let $p, q \in \mathcal{P}(A)$. Then $p \oplus q \sim_h q \oplus p$.

Proof. Consider the path

$$\gamma(t) = R(t) \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} R(-t) \quad \text{ where } \quad R(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t) & -\sin(\frac{\pi}{2}t) \\ \sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}$$

This is a path of projections in $\mathcal{P}_{2n}(A)$, such that $\gamma(0) = p \oplus q$ and $\gamma(1) = q \oplus p$. □

Now let us find relations between $\sim_h$, $\sim_u$ and $\sim$.

Proposition 1.2.2. Let $p \in \mathcal{P}(A)$ and p_t be a path of projections from p. Then there is a path of unitaries u_t such that $u_0 = 1$ and $u_t^* p u_t = p_t$.

Proof. First, suppose that $\forall t \in [0\,, 1]$, $\|p - p_t\| < 1$, and consider $x_t = pp_t + (p-1)(p_t - 1)$. Then, since $2p - 1$ is unitary, for all $t \in [0\,, 1]$, $\|x_t - 1\| < 1$, so x_t is invertible. We write its polar decomposition $x_t = u_t\,|x_t|$. Since $u_t \mapsto x_t$, $p_t \mapsto x_t$ and x_t are continuous, u_t is continuous. Direct computation shows that $x_t p_t = pp_t = x_t p_t$. Moreover, $p_t x_t^* x_t = p_t p = x_t^* x_t p_t$, which gives $x_t^* x_t p_t^2 = p_t^2 x_t^* x_t$. Hence $|x_t|\, p_t = p_t\, |x_t|$. Thus $pu_t = u_t p_t$, and so $\forall t \in [0\,, 1]\; u_t^* p u_t = p_t$. And $u_0 = x_0 = 1$. We get the wanted path.
Now consider the general case. Then, since p_t is uniformly continuous (because continuous on the compact $[0\,, 1]$), there exists a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\forall t \in [t_i\,, t_{i+1}]\,, \|p_{t_i} - p_t\| < 1$. Hence, we can apply the first point with p_{t_i} in the role of p, to get a path u_t^i on each interval $[t_i\,, t_{i+1}]$. Thus, by gluing the paths $u_t^{i+1} u_{t_i}^i$, we obtain the desired path of unitaries. □

By applying the previous proposition with $p_1 = q$, we get the following corollary.

Corollary 1.2.1. Let $p, q \in \mathcal{P}(A)$. If $p \sim_h q$, then $p \sim_u q$.

Lemma 1.2.1. Let $p, q \in \mathcal{P}(A)$. Let $v \in A$ such that $p = v^*v$ and $q = vv^*$. Then $qvp = qv = pv = v$.

Proof. We have $v^*vv^* = v$. Endeed :

$$\|v - vv^*v\|^2 = \|(v - vv^*v)^*(v - vv^*v)\| = \|(1-p)p(1-p)\| = 0$$

So $qvp = v$, $qvp = vv^*vv^*v = vp$ and $qvp = qv$. □

Proposition 1.2.3. Let $p, q \in \mathcal{P}(A)$ such that $p \sim q$. Then $p \sim_u q$ in $\mathcal{P}_2(A)$ and $p \sim_h q$ in $\mathcal{P}_4(A)$.

Proof. Define

$$U = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}$$

Then, it follows from the previous lemma that U is unitary, and we find that $U^*qU = p$. Thus, $p \sim_u q$ in $\mathcal{P}_2(A)$. Now define

$$V = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \in \mathcal{U}_4(A)$$

Then

$$V_t = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} R(-t) \begin{pmatrix} 1 & 0 \\ 0 & U^* \end{pmatrix} R(t)$$

defines a path of unitaries from $V_0 = V$ and $V_1 = I_4$. Note that $V^*qV = p$. Hence, it is a simple verification to see that $V_t^*qV_t$ is a path of projections from p to q. We get $p \sim_h q$ in $\mathcal{P}_4(A)$. ☐

Proposition 1.2.4. Let $p, q \in \mathcal{P}(A)$ such that $p \sim_u q$. Then $p \sim q$.

Proof. Let $u \in \mathcal{U}(A)$ such that $q = u^*pu$. Then $p \sim q$ by the partial isometry u^*p. ☐

All these results mean that if two projections are equivalent in one way, then they are in another way in $\mathcal{P}_n(A)$ for some sufficiently large n.

1.3 Tensor products of C^*-algebras

This subsection is mainly based on [Bla05] and [Mur90], where more details can be found. Denote by $A \otimes_{alg} B$ the algebraic tensor product of two vector spaces A and B over $\mathbb{C}$. Recall that it is defined in the following way. Consider the free vector space $F(A \times B)$ generated by $A \times B$. Then $A \otimes_{alg} B = F(A \times B)/\sim$ where $\sim$ is the equivalence relation defined on $A \times B$ by :

$$\begin{aligned}(a + a', b) &\sim (a, b) + (a', b) \\ (a, b + b') &\sim (a, b) + (a, b') \\ (\lambda a, b) &\sim (a, \lambda b) \sim \lambda(a, b)\end{aligned}$$

In fact, $A \otimes_{alg} B$ consists of all finite linear combinations of elements of the form $a \otimes b$. Recall that it satisfies the following universal property.

Proposition 1.3.1. Given three vector spaces A, B and C and a bilinear map $\varphi : A \times B \to C$, there exists a unique linear map $\tilde{\varphi}$ making the following diagram commute

$$\begin{tikzcd} A \times B \arrow[r, "\varphi"] \arrow[d] & C \\ A \otimes_{alg} B \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

Now we should endow it with a complete norm making it a C^*-algebra. But such a norm is generally not unique. A way of doing this will be with representations in Hilbert spaces. First, consider two Hilbert spaces H and K and endow $H \otimes_{alg} K$ with the inner product defined by

$$\langle x_1 \otimes y_1 \, , x_2 \otimes y_2 \rangle = \langle x_1 \, , x_2 \rangle_H \langle y_1 \, , y_2 \rangle_K$$

And define $H \bar{\otimes} K$ as the Hilbert space completion of $H \otimes_{alg} K$ for the norm defined by the inner product. Now, just before defining a first notion of tensor product of C^*-algebras, given two $*$-algebras A and B, endow $A \otimes_{alg} B$ with a multiplication and an involution, making it a $*$-algebra :

$$(a \otimes b) \cdot (a' \otimes b') = (aa' \otimes bb') \quad \text{and} \quad (a \otimes b)^* = a^* \otimes b^*$$

Now, let us define the minimal tensor product. Let A and B be two C^*-algebras. Consider their universal representations given by the GNS construction (which gives the proof of the Gelfand-Naimark theorem) $\rho_A : A \to \mathcal{B}(H_A)$ and $\rho_B : B \to \mathcal{B}(H_B)$. In particular, they are faithful. Then, for every operators $T \in \mathcal{B}(H_A)$ and $T' \in \mathcal{B}(H_B)$, by the universal property of the algebraic tensor product, we have an operator $T \otimes T' : H_A \otimes_{alg} H_B \to H_A \otimes_{alg} H_B \subset H_A \bar{\otimes} H_B$, which is bounded with respect to the norm on $H_A \bar{\otimes} H_B$ defined by the inner product above. Hence we can extend $T \otimes T'$ to a bounded operator

$T \otimes T' : H_A \bar{\otimes} H_B \to H_A \bar{\otimes} H_B$. This gives us the injective $*$-homomorphism $\varphi : A \otimes_{alg} B \to \mathcal{B}(H_A \bar{\otimes} H_B)$, given by $\varphi(a \otimes b) = \rho_A(a) \otimes \rho_B(b)$. And through it, we get a norm on $A \otimes_{alg} B$, from the operator norm of $\mathcal{B}(H_A \bar{\otimes} H_B)$. Finally, we take the completion of $A \otimes_{alg} B$ for this norm. We obtain a C^*-algebra called the minimal tensor product of A and B, denoted by $A \otimes_{min} B$.

Now we construct the maximal norm. We set

$$\|x\|_{max} = \sup_{\rho} \|\rho(x)\|$$

where ρ runs over all $*$-representations $\rho : A \otimes_{alg} B \to \mathcal{B}(H)$ such that for all $a \in A$ and $b \in B$, $\|\rho(a \otimes b)\| \leq \|a\| \|b\|$. The completion of $A \otimes_{alg} B$ in this norm is a C^*-algebra and we call it the maximal tensor product of A and B, denoted by $A \otimes_{max} B$.

There is a homomorphism $\pi_{A,B} : A \otimes_{max} B \to A \otimes_{min} B$, obtained in the following way. Consider the homomorphism $\pi_{A,B}$ given by the composition

$$A \otimes_{alg} B \xrightarrow{id} A \otimes_{alg} B \hookrightarrow A \otimes_{min} B$$

and consider the representation $\varphi : A \otimes_{min} B \to \mathcal{B}(H_A \bar{\otimes} H_B)$ of the above construction of $A \otimes_{min} B$. Then φ is a representation such that

$$\|\varphi(a \otimes b)\| = \|\rho_A(a) \otimes \rho_B(b)\| = \|\rho_A(a)\| \|\rho_B(b)\| = \|a\| \|b\|$$

Hence, by definition of $\|\cdot\|_{max}$,

$$\|\pi_{A,B}(a \otimes b)\|_{\min} = \|a \otimes b\| = \|\varphi(a \otimes b)\| \leq \|a \otimes b\|_{max}$$

So we can extend $\pi_{A,B}$ into an homomoprhism $\varphi_{A,B} : A \otimes_{max} B \to A \otimes_{min} B$.
We say that the C^*-algebra A is nuclear if $\pi_{A,B}$ is an isomorphism for every C^*-algebra B. In this case, there is only one C^*-completion of $A \otimes_{alg} B$, which we will simply denote by $A \otimes B$. One can find a proof of the following important statement in [Bla05].

Proposition 1.3.2. Every commutative C^*-algebra is nuclear.

Lemma 1.3.1. Let X be a locally compact Hausdorff space and let A be a C^*-algebra. Then $\operatorname{span}\{fa,\ f \in C_0(X),\ a \in A\}$ is dense in $C_0(X, A)$.

Proof. Denote by $X^+ = X \cup \{\infty\}$ the one-point compactification of X. Let

$$f \in C_0(X, A) \cong \{g \in C(X^+, A) \mid g(\infty) = 0\}$$

Let $\varepsilon > 0$. Then there exist $x_1, \ldots, x_n \in X^+$ such that we have the open cover $X^+ = \bigcup_{k=1}^n U_k$ where $U_k = \{x \in X^+ \mid \|f(x) - f(x_k)\|\}$. So we get a partition of unity, ie. there is continuous functions $h_1, \ldots, h_n : X^+ \to [0, 1]$ such that $h_1 + \cdots + h_n = 1$ and $\operatorname{supp}(h_k) \subset U_k$. We obtain that

$$\forall x \in X^+, \quad \left\| f(x) - \sum_{k=1}^n h_k(x) f(x_k) \right\| \leq \varepsilon$$

In particular, since $f(\infty) = 0$, $\|\sum_{k=1}^n h_k(\infty) f(x_k)\| = 0$. Let f_k be the restriction of $h_k - h_k(\infty)$ to X. Then $f_k \in C_0(X)$ and

$$\left\| f - \sum_{k=1}^n f_k f(x_k) \right\|_\infty \leq 2\varepsilon$$

□

Then, considering the map

$$\begin{array}{ccc} C_0(X) \times A & \longrightarrow & C_0(X, A) \\ (f, a) & \longmapsto & fa \end{array}$$

one can prove the following proposition. More details of the proof can be found in [Mur90].

Proposition 1.3.3. Given a locally compact Hausdorff space X and a C^*-algebra A, we have an isomorphism of C^*-algebras $C_0(X) \otimes A \cong C_0(X, A)$.

From now on, $\otimes$ will always denotes the minimal tensor product. Let us finish this section with two propositions we will need later on to prove Bott periodicity theorem. One can read proves in [Mur90].

Proposition 1.3.4. Let A, B, A' and B' be C^*-algebras. Let $\varphi : A \to B$ and $\psi : A' \to B'$ be two homomorphisms. Then there exists a unique homomorphism $\varphi \otimes \psi : A \otimes B \to A' \otimes B'$ such that $\forall (a, b) \in A \times B,\ \varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$.

Proposition 1.3.5. Let I, A, B and D be C^*-algebras such that $B \otimes_{alg} D$ has a unique C^*-norm and suppose that we have the short exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} B \longrightarrow 0$$

Then

$$0 \longrightarrow I \otimes D \xrightarrow{\iota \otimes id} A \otimes D \xrightarrow{\pi \otimes id} B \otimes D \longrightarrow 0$$

is a short exact sequence of C^*-algebras.

2 Definition of K-theory and main properties

In this section, we will construct the functors K_0, K_1, and K_{-n} for $n \geq 1$, which will associate to every C^*-algebra A an Abelian group. We will see their main properties, such as direct sum preserving, homotopy invariance, and the natural isomorphism between the functors K_1 and K_{-1}.

2.1 Preliminaries

First of all, we introduce some tools we will need, such as direct limit and the Grothendieck construction. But, before, let us recall a bit of category theory, which will help us to point out certain important properties of K-theory.

Definition 2.1.1. A category $\mathcal{C}$ is given by

- a class $\mathrm{Ob}(\mathcal{C})$ of objects,
- a class $\mathrm{Hom}(\mathcal{C})$ of morphisms, also called arrows, between the objects ; for two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, we denote by $\mathrm{Hom}(A, B)$ the class of morphisms from A to B, and for a morphism $f \in \mathrm{Hom}(A, B)$, we also write $f : A \to B$,
- for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$, a binary operation $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ called composition of morphisms.

such that :

- for every $f : A \to B$, $g : B \to C$ and $h : C \to D$, $h \circ (g \circ f) = (h \circ g) \circ f$,
- for every $X \in \mathrm{Ob}(\mathcal{C})$, there is a morphism $id_X : X \to X$ such that for every $f : A \to X$ and $g : X \to B$, $id_X \circ f = f$ and $g \circ id_X = g$.

Example 2.1.1. The category of groups **Grp** with morphisms of group, abelian groups **Ab**, topological spaces **Top** with continuous functions, C^*-algebras $\mathbf{C}^*$ with $*$-homomorphisms.

Definition 2.1.2. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a mapping which associate to each object A in $\mathcal{C}$ and object $F(A)$ in $\mathcal{D}$, and to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, such that :

- for every object X in $\mathcal{C}$, $F(id_X) = id_{F(X)}$,
- for every morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$.

A contravariant functor is defined the same way, except that it reverses the arrows and the composition.

Definition 2.1.3. Let F and G be two functor between categories $\mathcal{C}$ and $\mathcal{D}$. A natural transformation $\eta : F \to G$ is given by a morphism $\eta_A : F(A) \to G(A)$ for every $A \in \mathrm{Ob}(\mathcal{C})$, such that for every $A, B \in \mathrm{Ob}(\mathcal{C})$ and $f : A \to B$, the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

If η_A is an isomorphism for every $A \in \mathrm{Ob}(\mathcal{C})$, we call η an natural isomorphism.

Actually, a natural isomorphism between two functors tells us that they share the same properties and we can see them as the same functor.

Next notion we will need is direct limit of topological spaces. The description which follows is based on [NdK16]. If X be a topological space, we denote by $\pi_0(X)$ the set of path components of X.

Definition 2.1.4. Consider the sequence of maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

and note $f_{ij} = f_{j-1} \circ \cdots \circ f_i$ for $i < j$. We define the direct limit of the sequence as

$$\lim_{\substack{\rightarrow \\ i}} (X_i) = \left(\coprod_{i=0}^{+\infty} X_i \right) / \sim_c$$

where, for $x_i \in X_i$ and $x_j \in X_j$, $x_i \sim_c x_j$ if and only if there exists $k \geq i, j$ such that $f_{ik}(x_i) = f_{jk}(x_j)$.

$\sim_c$ is an equivalence relation. Furthermore, for each $k \in \mathbb{N}$, there is natural map $\iota_k : X_k \to \lim_i(X_i)$, given by the composition of the inclusion $X_k \to \coprod_{i=0}^{+\infty} X_i$ and the quotient map. Remark that, if the f_i are inclusions, then the direct limit is simply the union of the X_i. One relevant point about the direct limit is that it satisfies the following universal property.

Proposition 2.1.1. Given the commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[drr, bend right, "g_0"'] & X_1 \arrow[r, "f_1"] \arrow[dr, "g_1"] & X_2 \arrow[r, "f_0"] \arrow[d, "g_2"] & \cdots \arrow[r, "f_{k-1}"] & X_k \arrow[r, "f_k"] \arrow[dll, "g_k"] & \cdots \\
& & Y & & &
\end{tikzcd}$$

there exists a unique map $g : \lim_i X_i \to Y$ such that $\forall k \in \mathbb{N}$, $g_k = g \circ \iota_k$.

Proof. Define the map g as

$$\begin{array}{rccc} g: & \lim_i(X_i) & \longrightarrow & Y \\ & \iota_k(x_k) & \longmapsto & g_k(x_k) \end{array}$$

By writing down what it means, we see that this map is well defined (and so, is unique) and satisfies the equalities we want. □

If X_i are topological spaces and f_i are continuous, we can endow the direct limit with a topology : $F \subset \lim_i(X_i)$ is closed if and only if $\iota_k^{-1}(F)$ is closed in X_k for all $k \in \mathbb{N}$.

Lemma 2.1.1. Consider a sequence of continuous maps between topological spaces as above such that f_i are closed inclusions and that X_i are Hausdorff. Then

1. the maps $\iota_k : X_k \to \lim_i X_i$ are closed inclusions
2. any compact subset $K \subset \lim_i X_i$ lies in $\iota_k(X_k)$ for some $k \in \mathbb{N}$

Proof. First, let us prove the injectivity of ι_k. Let $x, y \in X_k$ such that $\iota_k(x) = \iota_k(y)$. Then, by definition of $\sim_c$, $f_{kl}(x) = f_{kl}(y)$ for some $l > k$. Thus, since f_i are all injective, $x = y$, and ι_k is injective. Now, let C be a closed subset of X_k. Then, for l>k, $\iota_k(C) = \iota_l \circ f_{kl}(C)$. Since ι_l is injective and f_i are closed inclusions, $\iota_l^{-1}(\iota_k(C)) = f_{kl}(C)$ is a closed subset of X_l. For $l < k$, by continuity of f_{lk}, $\iota_l^{-1}(\iota_k(C)) = f_{lk}^{-1}(C)$ is closed. Obviously, $\iota_k^{-1}(\iota_k(C)) = C$ is closed. Thus, $\iota_k(C)$ is closed in $\lim_i X_i$, and so ι_k is a closed inclusion.
Let us now prove the second item. Set $X = \lim_i X_i$. For each $k \in \mathbb{N}$, define $Y_k = X_k \setminus f_{k-1}(X_{k_1})$. When $\iota_k(Y_k) \cap K \neq \varnothing$, take $y_k \in \iota_k(Y_k) \cap K$ dans consider the set Y of all these y_k. Because of the injectivity of the f_i and by definition of Y_k, $Y \cap \iota_k(X_k)$ is finite. Hence $\iota_k^{-1}(Y)$ is a finite subset of the Hausdorff space X_k, and so $\iota_k^{-1}(Y)$ is closed. Thus Y is a closed subset of X, which is compact : so is Y. Let $y \in Y$. By the same reasoning, we find that $Y_y = Y \setminus y$ is closed. Then $X \setminus Y_y$ is open, and so $y = Y \cap (X \setminus Y_y)$ is open in Y. Hence, Y is discrete. Since it is compact, Y is finite. This implies that $\forall k > n$, $\iota_k(Y_k) \cap K = \varnothing$ for some n. Thus $K \in \iota_m(X_m)$ for some m. □

Corollary 2.1.1. In the situation of the previous lemma, the natural map $\lim_i \pi_0(X_i) \to \pi_0(\lim_i X_i)$ is a bijection.

Proof. Denote by Φ this map, and by [.] the equivalence classes for π_0. We consider the sequence of maps

$$\pi_0(X_0) \xrightarrow{(f_0)_0} \pi_0(X_1) \xrightarrow{(f_1)_0} \pi_0(X_2) \xrightarrow{(f_2)_0} \cdots$$

where $(f_i)_0$ are defined by $(f_i)_0([x_i]) = [f_i(x_i)]$ (well-defined since f_i are continuous) and where we denote by j_k the map $\pi_0(X_k) \to \lim_i \pi_0(X_i)$. Note that $(f_{kl})_0 = (f_{l-1})_0 \circ \cdots \circ (f_k)_0$, by functoriality of π_0. Consider the map Φ defined by

$$\Phi([x]) = j_k([x_k]) \quad \text{where } x = \iota_k(x_k) \text{ for some } k \text{ and some } x_k \in X_k$$

First of all, we have to show that this is well-defined, ie. that $\Phi(x)$ does not depend on neither the choice of k nor the choice of x. If $x = \iota_k(x_k) = \iota_l(x_l)$ for some $k < l$, then, since f_i are injective, $x_l = f_{kl}(x_k)$, and so $(k_{kl})_0([x_k]) = [f_{kl}(x_k)] = [x_l]$, which implies $j_k([x_k]) = j_l([x_l])$. Now, let $x, y \in \lim_i X_i$ such that $[x] = [y]$. Then there is a path γ from x to y. Since the image of γ is compact, it lies in $\iota_k(X_k)$ for some k, by the previous lemma. In particular, there exists $x_k, y_k \in X_k$ such that $x = \iota_k(x_k)$ and $y = \iota_k(y_k)$. Hence,

since ι_k is injective, $\iota_k^{-1}(\gamma)$ defines a path connecting x_k and y_k. Thus $[x_k] = [y_k]$ and $j_k([x_k]) = j_k([y_k])$. It shows that Φ is well-defined.
Let us prove that it is a bijection. Let $[x], [y] \in \pi_0(\lim_i X_i)$ such that $\Phi([x]) = \Phi([y])$. Then $x = \iota_k(x_k)$ and $y = \iota_l(y_l)$ for some $k < l$. So $j_k([x_k]) = j_l([y_l])$, and, by injectivity of f_i, $(f_{kl})_0([x_k]) = [y_l]$, ie. $[f_{kl}(x_k)] = [y_l]$. Then there is a path α connecting $f_{kl}(x_k)$ and y_l. Hence, $\iota_l \circ \alpha$ is a path between $\iota_k(x_k)$ and $\iota_l(y_l)$ in $\lim_i X_i$. Thus $[x] = [\iota_k(x_k)] = [\iota_l(y_l)] = [y]$. For surjectivity, if $a \in \pi_0(\lim_i X_i)$, then $a = j_k([x_k])$ for some k and $x_k \in X_k$, and $a = \Phi([\iota_k(x_k)])$. $\square$

Now we introduce another tool we will need in the following : the Grothendieck construction. It associates to an abelian monoid an abelian group. Let $(S, +)$ be an abelian monoid. Define

$$G(S) = (S \times S) \,/ \sim_g$$

where $\sim_g$ is the equivalence relation on $S \times S$ defined by $(a, b) \sim_g (c, d)$ if and only if there exists $e \in S$ such that $a + d + e = b + c + e$. On can think of (a, b) as the formal difference $a - b$. We have the monoid map $i_S : S \to G(S)$ given by $i_S(a) = (a, 0)$. The operation $[(a, b)] + [(c, d)] = [(a + c, b + d)]$ on $S \times S$ turns $G(S)$ into an abelian group. It has the following universal property, whose proof is a simple verification.

Proposition 2.1.2. Let S be an abelian monoid. For any monoid map f from S to an abelian group H, there is a unique group homomorphism $\tilde{f} : G(S) \to H$ such that the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "i_S"'] & H \\ G(S) \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Moreover $\tilde{f}$ is given by $\tilde{f}([(x, y)]) = f(x) - f(y)$.

Given two abelian monoids S and T, and a monoid map $\varphi : A \to B$, one can associate a group homomorphism $G(\varphi) : G(S) \to G(T)$, by setting $G\varphi([x, y]) = [(\varphi(x), \varphi(y))]$. It is easy to check that this is well-defined and is a homomorphism. This construction turns G into a functor from the category **cMon** of abelian monoids to the category **Ab** of abelian groups.

Example 2.1.2. The Grothendieck construction is exactly what we use to get $\mathbb{Z} = G(\mathbb{N})$ from $\mathbb{N}$.

2.2 The functor K_0

Now we are ready to define the group K_0 for a unital C^*-algebra. The following is mainly based on [Bla86], [MM15] and [NdK16]. We will denote by $\mathcal{M}_\infty(A)$ the set $\bigcup_{n=1}^{+\infty} \mathcal{M}_n(A)$. Let A be a unital C^*-algebra. We have the sequence of inclusions of Hausdorff spaces

$$\mathcal{P}_1(A) \subset \mathcal{P}_2(A) \subset \mathcal{P}_3(A) \subset \cdots$$

where each inclusion is the embedding $a \mapsto \mathrm{diag}(a, 0)$. π_0 is a functor from the category **Top** of topological spaces to the category **Set** of sets. Hence, a map $f : X \to Y$ between two topological spaces X and Y, induces the map $\pi_0 f$, often still denoted by f, defined by $\pi_0 f([x]) = [f(x)]$. So we have the sequence of maps

$$\pi_0\mathcal{P}_1(A) \to \pi_0\mathcal{P}_2(A) \to \pi_0\mathcal{P}_3(A) \to \cdots$$

and we define

$$V(A) = \lim_i \pi_0(\mathcal{P}_i(A))$$

From corollary 2.1.1, we have the bijection $V(A) \simeq \pi_0(\lim_i \mathcal{P}_i(A))$. It means that each element of $V(A)$ is of the form $[p]$ where $p \in \mathcal{P}_n(A)$ for some n (also seen as an infinite matrix with p in the upper-left corner and zeros otherwise) and where $[p]$ denotes the path component of the image of p in the direct limit. So we can give $V(A)$ a monoid structure, by setting $[p] + [q] = [p \oplus q]$.

Proposition 2.2.1. With this addition, $V(A)$ is an abelian monoid.

Proof. Firstly, this operation is well defined, because, if $p \sim_h p'$ by $\gamma_1(t)$ and $q \sim_h q'$ by $\gamma_2(t)$, then $p \oplus q \sim_h p' \oplus q'$ by $\gamma_1(t) \oplus \gamma_2(t)$. It is a simple verification to see that it is a monoid, with $[0]$ as the identity element. Commutativity comes from the fact that $p \oplus q \sim_h q \oplus p$, as we saw in proposition 1.2.1. $\square$

Finally, we define

$$K_0(A) = G(V(A))$$

In fact, since then $([p],[q]) \sim_g ([p'],[q'])$ implies $p \oplus q' \oplus r \sim_h p' \oplus q \oplus r$ for some $r \in \mathcal{P}_\infty(A)$, we can see any element of the group $K_0(A)$ as a formal difference $[p] - [q]$ where p and q are projections in matrices over A and where, in virtue of the subsection 1.2, $[.]$ denotes the equivalent class under $\sim_h$, $\sim_u$ or $\sim$ in $\mathcal{P}_\infty(A)$. Given two unital C^*-algebras A and B, and a homomorphism $\varphi : A \to B$, one can associate a group homomorphism $K_0(\varphi)$, often denoted by φ_*, by $\varphi_*([p] - [q]) = [\varphi(p)] - [\varphi(q)]$, where φ extends entrywisely to matrices over A. This construction turns K_0 into a functor from the category $\mathbf{uC}^*$ of unital C^*-algebras to the category $\mathbf{Ab}$ of abelian groups. Next proposition will help us to draw a more simple picture of $K_0(A)$.

Proposition 2.2.2. Let $p, q \in \mathcal{P}(A)$. If $pq = qp = 0$, then $p \oplus q \sim_h (p+q) \oplus 0$.

Proof. For $t \in [0\,,1]$, consider

$$\gamma(t) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} + R(t) \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} R(-t)$$

Then, by a simple computation, we see that γ is a path of projections in $\mathcal{P}_2(A)$ connecting $p \oplus q$ and $(p+q) \oplus 0$. □

From now on, we denote by ℓ_n the identity matrix 1_n of $\mathcal{M}_n(A)$ viewed in $\mathcal{M}_m(A)$ for $m \leq n$ as well as in $\mathcal{M}_\infty(A)$: $\ell_n = 1_n \oplus 0_{m-n}$. Note that $\ell_0 = 0$ in $\mathcal{M}_\infty(A)$.

Corollary 2.2.1. Let $x \in K_0(A)$. Then $x = [p] - [\ell_k]$ for some projection $p \in \mathcal{P}_\infty(A)$ and some $k \in \mathbb{N}$.

Proof. We know that $x = [p] - [q]$ for some projections $p, q \in \mathcal{P}_n(A)$ and some n. Then $1_n - q$ is a projection and

$$x = [p] + [1_n - q] - ([1_n - q] - [q]) = [p \oplus (1_n - q)] - [(1_n - q) \oplus q] = [p \oplus (1_n - q)] - [\ell_n]$$

since $(1_n - q)q = 0$. □

The following proposition states that K_0 preserves direct sum.

Proposition 2.2.3. $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$

Proof. Note that $\mathcal{M}_n(A \oplus B) \cong \mathcal{M}_n(A) \oplus \mathcal{M}_n(B)$. Consider the two projection maps $pr_A : A \oplus B \to A$, $pr_B : A \oplus B \to B$. By functoriality of K_0, they induce homomorphisms $(pr_A)_* : K_0(A \oplus B) \to K_0(A)$ and $(pr_B)_* : K_0(A \oplus B) \to K_0(B)$. Then consider the homomorphism $\Phi = (pr_A)_* \oplus (pr_B)_* : K_0(A \oplus B) \to K_0(A) \oplus K_0(B)$. Let $[p] - [p] \in K_0(A \oplus B)$ such that $\Phi([p] - [q]) = 0$, where $p, q \in \mathcal{P}_\infty(A \oplus B)$. Write $p = (p_A, p_B)$ and $q = (q_A, q_B)$ where $p_A, q_A \in \mathcal{P}_\infty(A)$ and $p_B, q_B \in \mathcal{P}_\infty(B)$. Then $[p_A] = [q_A]$ and $[p_B] = [q_B]$. So there is partial isometries $v_A \in \mathcal{M}_\infty(A)$ and $v_B \in \mathcal{M}_\infty(B)$ such that $p_A = v_A^* v_A$, $q_A = v_A v_A^*$ and $p_B = v_B^* v_B$, $q_B = v_B v_B^*$. Hence, $v = (v_A, v_B) \in \mathcal{M}_\infty(A \oplus B)$ is a partial isometry between p and q. It shows that Φ is injective. For the surjectivity, if $([p_A] - [q_A], [p_B] - [q_B]) \in K_0(A) \oplus K_0(B)$, then it is equal to $\varphi([(p_A, p_B)] - [(q_A, q_B)])$ □

A is not necessary unital. So we want to define a compatible K_0 for non-unital C^*-algebras. Recall from the definition of the unitalization A^+ of A (see 1.1) that we have the short exact sequence of C^*-algebras

$$0 \longrightarrow A \longrightarrow A^+ \longrightarrow \mathbb{C} \longrightarrow 0$$

which splits by $\lambda \to (0, \lambda)$ (namely a map whose right composition with the third arrow gives the identity on $\mathbb{C}$). Since K_0 is a functor, the third arrow induces on K-theory a map $K_0(A^+) \to K_0(\mathbb{C})$. We define

$$K_0(A) = \ker(K_0(A^+) \to K_0(\mathbb{C}))$$

Note that this definition is compatible with the previous one in the case of unital C^*-algebras. Indeed, suppose A is unital. Then $A \oplus \mathbb{C} \cong A^+$ as C^*-algebras, by the isomorphism $(a, \lambda) \mapsto (a - \lambda, \lambda)$. Hence, by proposition 2.2.3, $K_0(A^+) \cong K_0(A) \oplus K_0(\mathbb{C})$, and so $\ker(K_0(A^+) \to K_0(\mathbb{C})) \cong K_0(A)$. Note that this new definition extends K_0 to a functor from $\mathbf{C}^*$ to $\mathbf{Ab}$ and that 2.2.3 still holds for non-unital C^*-algebras (split-exactness of K_0 (see subsection 2.4) will give us a proof later on). Another important fact about K_0 is that it is homotopy invariant.

Proposition 2.2.4. Let A and B two C^*-algebras, and $\varphi, \psi : A \to B$ two homotopic homomorphisms. Then φ and ψ induce the same map $\varphi_* = \psi_* : K_0(A) \to K_0(B)$.

Proof. Denote by $F : A \to C([0,1], B)$ the homotopy between φ and ψ : $ev_0 \circ F = \varphi$ and $ev_1 \circ F = \psi$. Since unitalization, $\mathcal{M}_n$ and $\mathcal{P}$ are functors, it induces a map $F_n^+ : \mathcal{P}_n(A^+) \to \mathcal{P}_n(C([0,1], B^+)$. Note that, if $p \in \mathcal{P}_n(A^+)$ for some n, then $ev_t \circ F_n^+(p)$ is a path from $\varphi_n^+(p)$ to $\psi_n^+(p)$. Let $[p] - [q] \in K_0(A^+)$, where $p, q \in \mathcal{P}_n(A^+)$ for some n. Hence

$$\varphi_*^+([p] - [q]) = [\varphi_n^+(p)] - [\varphi_n^+(q)] = [\psi_n^+(p)] - [\psi_n^+(q)] = \psi_*^+([p] - [q])$$

So $\varphi_*^+ = \psi_*^+$. Thus, their restrictions φ_* and ψ_* to $\ker(K_0(A^+) \to K_0(\mathbb{C}))$ are equal. □

Corollary 2.2.2. Given two C^*-algebras A and B, if $A \approx B$, then $K_0(A) \cong K_0(B)$.

Proof. Let $f : A \to B$ and $g : B \to A$ two homotopy equivalences which implement $A \approx B$. Then, by definition, $f \circ g$ and $g \circ f$ are respectively homotopic to id_B and id_A. Hence they induces homomorphisms $f_* \circ g_* = id_{K_0(B)}$ $g_* \circ f_* = id_{K_0(A)}$. Thus $f : K_0(A) \to K_0(B)$ is an isomoprhism. □

Now let us compute our first K-group : $K_0(\mathbb{C})$.

Lemma 2.2.1. For any $n \geq 1$, the map $\mathrm{tr} : \pi_0\mathcal{P}_n(\mathbb{C}) \to [\![0\,, n]\!]$ induced by the trace tr of matrices is a bijection.

Proof. First we need to show that this map is well defined, ie. that the trace is constant on the path connected components of $\mathcal{P}_n(\mathbb{C})$ and that its image is contained in $[\![0\,, 1]\!]$. If $p, q \in \mathcal{P}_n(\mathbb{C})$ are in the same path component, then they are unitarily equivalent by corollary 1.2.1, so they have the same trace. Now let $p \in \mathcal{P}_n(\mathbb{C})$. Then, since p is idempotent, its Jordan normal form is ℓ_k for some $k \leq n$. And so $0 \leq \mathrm{tr}(p) \leq n$. Let us prove that this map is a bijection. If two projections p, q have the same trace, then they share the same Jordan normal form ℓ_k for some $k \leq n$: $p = z\ell_k z^{-1}$ and $q = w\ell_k w^{-1}$ for some $z, w \in \mathrm{GL}_n(\mathbb{C})$. Since $\mathrm{GL}_n(\mathbb{C})$ is path connected, there is a path of invertible x_t from z to w. Hence $x_t\ell_k x_t^{-1}$ is a path of projections connecting p and q, which shows injectivity. For surjectivity, every $k \in [\![0\,, n]\!]$ is the image of the path component of the projection ℓ_k. □

Proposition 2.2.5. $K_0(\mathbb{C}) \cong \mathbb{Z}$ and $[\ell_1]$ is a generator.

Proof. Following the previous lemma, it is easy to see that the diagram

$$\begin{tikzcd}
\pi_0\mathcal{P}_1(\mathbb{C}) \arrow[r] \arrow[drr, bend right, "\mathrm{tr}"'] & \pi_0\mathcal{P}_2(\mathbb{C}) \arrow[r] \arrow[dr, "\mathrm{tr}"] & \pi_0\mathcal{P}_3(\mathbb{C}) \arrow[r] \arrow[d, "\mathrm{tr}"] & \cdots \arrow[r] & \pi_0\mathcal{P}_k(\mathbb{C}) \arrow[r] \arrow[dll, "\mathrm{tr}"] & \cdots \\
 & & \mathbb{N} & & &
\end{tikzcd}$$

commutes. Hence the universal property of the direct limit (see proposition 2.1.1) gives us a map $\mathrm{tr} : \lim_i \pi_0(\mathcal{P}_i(\mathbb{C})) \to \mathbb{N}$, which is injective since the induced map tr are by the previous lemma. And it is obviously surjective. Furthermore, since $\mathrm{tr}(\ell_k \oplus \ell_l) = k + l$, tr is a map of monoids from $V(\mathbb{C})$ to $\mathbb{N}$. So, by functoriality of the Grothendieck contruction, it extends to an isomorphism of groups $\mathbb{K}_0(\mathbb{C}) \to \mathbb{Z}$. Finally, $[\ell_1]$ generates $V(\mathbb{C})$ and so generates $K_0(\mathbb{C})$. □

2.3 The functor K_1 and higher K-functors

In this subsection, we define the group $K_1(A)$ for a C^*-algebra A, show its main properties and how K_1 and K_0 are related. We will see that K_1 is a functor $\mathbf{C}^* \to \mathbf{Ab}$, preserves direct sums and is an homotopy invariant. The definition which follows comes from [RLL00], and some proofs come from [Bla86].

Let A be a C^*-algebra. Let us construct $K_1(A)$. We have the sequence of inclusions

$$\mathcal{U}_1(A^+) \subset \mathcal{U}_2(A^+) \subset \mathcal{U}_3(A^+) \subset \cdots$$

where each inclusion is the embedding $u \mapsto \mathrm{diag}(u, 1)$. So, consider on $\mathcal{U}_\infty(A^+) = \bigcup_{n=1}^{+\infty} \mathcal{U}_n(A^+)$ the equivalence relation $\sim_1$ defined by : $u \sim_1 v$ if and only if there is a path of unitaries connecting u and v in $\mathcal{U}_n(A^+)$ for some n. The proof that it defines a equivalence relation is a simple verification. Note that we will often see any matrix $a \in \mathcal{M}_n(A)$, as a matrix in $\mathcal{M}_m(A)$, still denoted by a, under the embedding above, for all $m \geq n$.

Definition 2.3.1. $K_1(A) = \mathcal{U}_\infty(A^+)/\sim_1$

Now let us define a group structure on $K_1(A)$ by defining on $K_1(A)$ the binary operation $[u]+[v]=[uv]$. This operation is well-defined (since the product of paths from u to u' and from v to v' respectively is a path connecting uv and $u'v'$), associative, and has the identity $0=[1]$, where $1=1_n \in \mathcal{M}_\infty(A^+)$ for any $n \geq 1$. It leads us the following proposition.

Proposition 2.3.1. This operation turns $K_1(A)$ into an abelian group, and

$$\forall [u],[v] \in K_1(A), \quad [u]+[v]=[uv]=[u\oplus v]=[vu]$$

Proof. We just need to show the equalities, for commutativity of the operation. Let $u, v \in \mathcal{U}_n(A^+)$ for some n. Define

$$U(t) = R(t)\begin{pmatrix} v & 0 \\ 0 & 1\end{pmatrix} R(-t) \in \mathcal{U}_{2n}(A^+)$$

This is a path of unitaries from $v\oplus 1_n$ to $1_n \oplus v$. Hence uU is path of unitaries from $uv \oplus 1_n$ to $u\oplus v$ and Uu is a path of unitaries from vu to $u \oplus v$. Thus $[uv]=[u\oplus v]=[vu]$. ☐

Since $U_n(\mathbb{C})$ is path connected for any $n\geq 1$, we can directly deduce the following proposition.

Proposition 2.3.2. $K_1(\mathbb{C})=0$

Since unitalization and $\mathcal{M}_n$ are functors, it is easy to see that K_1 is a functor $\mathbf{C}^* \to \mathbf{Ab}$. For a homomorphism $\varphi : A \to B$, like for K_0, we will also denote by φ_* the induced map $K_1(\varphi)$, since context will always be clear. Now let us mention a first link between K_1 and K_0. First, introduce the suspension of A.

Definition 2.3.2. The suspension of A, denoted by $\mathcal{S}A$ is defined by $\mathcal{S}A = C_0(\mathbb{R}, A)$, the set of continuous functions from $\mathbb{R}$ to A vanishing at infinity.

Remark 2.3.1. Note that $\mathcal{S}A$ is a non-unital C^*-algebra. Moreover, it is easy to see that

$$\mathcal{S}A \cong \{f\in C([0\,,1],A) \mid f(0)=f(1)=0\} \cong \left\{f\in C(S^1,A) \;\middle|\; f(1)=0\right\}$$

since S^1 is homeomorphic to the one-point compactification $\mathbb{R}\cup\{\infty\}$ of $\mathbb{R}$.

If we have a homomorphism of C^*-algebras $\varphi : A \to B$, then we get the induced $*$-homomorphism $\mathcal{S}\varphi : \mathcal{S}A \to \mathcal{S}B$ defined by $\mathcal{S}\varphi(f) = \varphi \circ f$. It turns $\mathcal{S}$ into a functor $\mathbf{C}^* \to \mathbf{C}^*$. We denote by $\mathcal{S}^n$ the composition of $\mathcal{S}$ n times. Note that $\mathcal{S}^n A \cong C_0(\mathbb{R}^n, A)$. Then, for every $n\in\mathbb{N}$, we define the functor $K_{-n} = K_0 \circ \mathcal{S}^n$. We can see that $\mathcal{S}$ preserves homotopy. Indeed, if $\Phi : A \to C([0\,,1],B)$ is an homotopy between two homomorphisms φ and ψ, then the following map is an homotopy between $\mathcal{S}\varphi$ and $\mathcal{S}\psi$:

$$\begin{array}{rccc} \mathcal{S}\Phi : & \mathcal{S}A & \longrightarrow & C([0\,,1],\mathcal{S}B) \\ & f & \longmapsto & \Phi\circ f \end{array}$$

Proposition 2.3.3. There is a natural isomorphism $K_1 \cong K_{-1}$.

Proof. We have to find an isomorphism θ_A for each C^*-algebras A, such that, for every homomorphism $\varphi : A \to B$ between C^*-algebras A and B, the following diagram commutes :

$$\begin{tikzcd} K_1(A) \arrow[r, "\varphi_*"] \arrow[d, "\theta_A"] & K_1(B) \arrow[d, "\theta_B"] \\ K_0(\mathcal{S}A) \arrow[r, "\mathcal{S}\varphi_*"] & K_0(\mathcal{S}B) \end{tikzcd}$$

First, let us define θ_A. Let $[u]\in K_1(A)$, where $u \in \mathcal{U}_n(A^+)$. We have $u\oplus u^* \sim_1 uu^* = 1_{2n}$ for n large enough (see proposition 2.3.1), by some path of unitaries z_t from 1_{2n} to $u\oplus u^*$. Set $f_t = z_t \ell_n z_t^*$. Then, noting that

$$(\mathcal{S}A)^+ \cong \left\{f\in C([0\,,1],A^+) \;\middle|\; f(0)=f(1)=\lambda 1 \text{ and } f(t) = a(t)+\lambda 1 \text{ for } \lambda\in\mathbb{C},\, a(t)\in A\right\}$$

we see that $f\in\mathcal{P}_{2n}((\mathcal{S}A)^+)$. Finally define $\theta_A([u]) = [f]-[\ell_n]$. Since $ev_0(f)=\ell_n$, we have $\theta_A([u]) \in K_0(\mathcal{S}A) = \ker(ev_0)_*$, with $(\mathcal{S}A)^+$ seen as above.

We have to prove that it is well defined. Let $[u] = [v] \in K_1(A)$, where $u, v \in \mathcal{U}_n(A^+)$. Then $u \sim_1 v$, and so $v^*u \sim_1 1_n$ and $1_n \sim_1 vu^*$, by paths of unitaries a_t from 1_n to v^*u and b_t from 1_n to vu^* respectively. Now, let z_t and w_t be paths of unitaries respectively from 1_{2n} to $u \oplus u^*$ and from 1_{2n} to $v \oplus v^*$, as in the definition of θ_A above. And set $f_t = z_t \ell_n z_t^*$ and $g_t = w_t \ell_n w_t^*$. Then, setting $x_t = w_t(a_t \oplus b_t)z_t^* \in \mathcal{U}_{2n}(A^+)$, we get $x \in \mathcal{U}_{2n}((\mathcal{S}A)^+)$ and $x_t f_t x_t^* = g_t$, so $f \sim_u g$ in $\mathcal{P}_{2n}((\mathcal{S}A)^+)$. Thus $[f] = [g]$, and so $\theta_A([u]) = [f] - [\ell_n] = [g] - [\ell_n] = \theta_A([v])$. Now, θ_A is easily seen to be a homomorphism between $K_1(A)$ and $K_{-1}(A)$ and it is a simple verification to show that the diagram above commutes.

Let us prove the injectivity. Let $[u] \in K_1(A)$ such that $\theta_A([u]) = [1_n] = 0$, and let z, f be as above. Then $[f] = [\ell_n]$, so there is a unitary $x \in \mathcal{U}_{2n}((\mathcal{S}A)^+)$ such that $x_t f_t x_t^* = \ell_n$. So, since $x_0 \in \mathcal{U}_{2n}(\mathbb{C})$, we may assume that $x_0 = 1_{2n}$ (by conjugating the previous equality by x_0 for instance). Note that $x_t z_t \ell_n z_t^* x_t^* = \ell_n$, ie. $x_t z_t$ commutes with ℓ_n, and so $x_t z_t$ must be of the form

$$x_t z_t = \begin{pmatrix} c_t & 0 \\ 0 & d_t \end{pmatrix}$$

where $c, d \in \mathcal{U}_n((\mathcal{S}A)^+)$, and, since $x_1 = x_0 = 1_{2n}$, we have

$$\begin{pmatrix} c_0 & 0 \\ 0 & d_0 \end{pmatrix} = 1_{2n} \quad \text{and} \quad \begin{pmatrix} c_1 & 0 \\ 0 & d_1 \end{pmatrix} = x_1(u \oplus u^*) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$$

Hence, c_t is a path of unitaries from 1_n to u, which shows that $1_n \sim_1 u$, and so $[u] = [1] = 0$.

For surjectivity, let $[f] - [\ell_k] \in K_0(\mathcal{S}A)$. So f is a path of projections in $\mathcal{P}_n(A^+)$ (where we may suppose $n \geq 2k$) such that $f_0 = f_1 \in \mathbb{C}$ and $f_t \equiv f_0 \mod A$. And then, in $K_0(\mathbb{C})$, $[f_0] = [\ell_k]$, so $f_0 = \ell_k$ up to conjugacy by a unitary and we may assume that $f_1 = f_0 = \ell_k$. Furthermore, since $[f_t] = [l_k]$ in $K_0(A^+)$ for all t, $f_t \equiv \ell_k \mod A$. Now, by proposition 1.2.2, there is a path of unitaries w_t in $\mathcal{U}_n((\mathcal{S}A)^+)$ such that $w_0 = 1_n$ and $f_t = w_t f_0 w_t^* = w_t \ell_k w_t^*$. Then $w_1 = w_0 = 1_n$ and $w_t \equiv 1_n \mod A$. Since $f_1 = \ell_k$, w_t commutes with ℓ_k, and so w_t must be of the form

$$w_t = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

where $u \in \mathcal{U}_k(A^+)$ and $v \in \mathcal{U}_{n-k}(A^+)$. By properties of $\sim_1$ (see proposition 2.3.1), we get

$$\begin{pmatrix} v^* & 0 \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1_{n-k} \end{pmatrix} \sim_1 \begin{pmatrix} 1_k & 0 \\ 0 & v^* \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1_{n-k} \end{pmatrix} = \begin{pmatrix} u^* & 0 \\ 0 & v^* \end{pmatrix} = w_1^* = 1_n$$

ie. $1_{n-k} \sim_1 v^*(u^* \oplus 1_{n-2k})$. So there is a path of unitaries a_t in $\mathcal{U}_{n-k}(A^+)$ (expand n if necessary) from 1_{n-k} to $v^*(u^* \oplus 1_{n-2k})$. Furthermore, by using the properties of $\sim_1$, it is easy to see that

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u^* & 0 \\ 0 & 0 & 1_{n-2k} \end{pmatrix} \sim_1 1_n$$

Then, let z_t be a path of unitaries from 1_n to $u \oplus u^* \oplus 1_{n-2k}$ in $\mathcal{U}_n(A^+)$. Set $g_t = z_t \ell_k z_t^*$, which defines an element of $\mathcal{P}_n((\mathcal{S}A)^+)$ (as in the definition of θ_A above), and so $\theta_A(u) = [g] - [\ell_k]$. And set $x_t = w_t(1_k \oplus a_t)z_t^*$, which defines a element of $\mathcal{U}_n((\mathcal{S}A)^+)$ such that $x_t g_t x_t^* = f_t$, and so $[f] = [g]$ in $K_0(\mathcal{S}A)$. Thus, we have $[f] - [\ell_k] = [g] - [\ell_k] = \theta_A([u])$. □

This last proposition tells us that K_1 shares many properties of K_0, actually those that $\mathcal{S}$ preserves : homotopy invariance, direct sum (as we will prove later on). We can already deduce the following two statements.

Proposition 2.3.4. Let A and B two C^*-algebras, and $\varphi, \psi : A \to B$ two homotopic homomorphisms. Then, for each $n \leq 1$, φ and ψ induce the same map $\varphi_* = \psi_* : K_n(A) \to K_n(B)$.

Corollary 2.3.1. Given two C^*-algebras A and B, if $A \approx B$, then $K_n(A) \cong K_n(B)$ for every $n \leq 1$.

2.4 Half and split exactness

Here we prove a relevant property and a first important step toward Bott periodicity theorem : half-exactness of the functors K_n for $n \leq 1$. Then split-exactness will extend this result. First let us look at this property for K_0. The following proofs are based on [Bla86] and [Mur90].

Lemma 2.4.1. Let $\varphi : A \to B$ be a surjective homomorphism between two unital C^*-algebras A and B. Then for any unitary $u \in B$, the matrix $u \oplus u^*$ is in the image of the map $\mathcal{U}_2(A) \to \mathcal{U}_2(B)$ induced by φ.

Proof. We have

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^* & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since φ is surjective, there are lifts v of u and w of u^*. Then

$$\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a product of unitaries (so is unitary) and a lift of $u \oplus u^*$. $\square$

Proposition 2.4.1. Let

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Then the induced sequence

$$K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(B)$$

is exact. In other words, the functor K_0 is half exact.

Proof. Denote by i the second arrow of the short exact sequence of C^*-algebras, and by π the third one. First, by functoriality of K_0, we have the induced sequence, and $\pi_* \circ i_* = K_0(\pi) \circ K_0(i) = K_0(\pi \circ i) = K_0(0) = 0$. Then $\mathrm{im}(\pi_* \circ i_*) = 0$. Hence $\mathrm{im}(i_*) \subset \ker(\pi_*)$. Let us show the other inclusion. Note that unitalization gives us the short exact sequence

$$0 \longrightarrow I^+ \xrightarrow{\iota^+} A^+ \xrightarrow{\pi^+} B^+ \longrightarrow 0$$

Let $[p] - [\ell_k] \in \ker(\pi_*) \subset K_0(A) \subset K_0(A^+)$ where $p \in \mathcal{P}_n(A^+)$ and $k \leq n$ (see corollary 2.2.1). Then $[\pi^+(p)] = [\pi^+(\ell_k)] = [\ell_k]$. Hence, there is a unitary $u \in \mathcal{U}_n(B^+)$ such that $u\pi^+(p)u^* = \ell_k$, and so

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \pi^+(p) & 0 \\ 0 & 0_n \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} \ell_k & 0 \\ 0 & 0_n \end{pmatrix}$$

Then, by lemma 2.4.1, there is $v \in \mathcal{U}_{2n}(A^+)$ such that $\pi^+(v) = u \oplus u^*$. Consider $q = v(p \oplus 0_n)v^* \in \mathcal{P}_{2n}(A^+)$. We have $\pi^+(q) = \ell_k$. Hence $q - \ell_k \in \ker(\pi^+) = \mathrm{im}(\iota^+)$. Furthermore $\ell_k = \iota^+(\ell_k)$, so $q \in \mathrm{im}(\iota^+)$, and $[q] - [\ell_k] \in \mathrm{im}(\iota_*^+)$. Thus, $[p] - [\ell_k] = [v(p \oplus 0_n)v^*] - [\ell_k] = [q] - [\ell_k] \in \mathrm{im}(\iota_*^+)$. Now, by reasoning on the arrows of the following induced commutative diagram with vertical short half exact sequences (by definition of K_0 for non-unital C^*-algebras)

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(B) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(I^+) & \xrightarrow{\iota_*^+} & K_0(A^+) & \xrightarrow{\pi_*^+} & K_0(B^+) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(\mathbb{C}) & = & K_0(\mathbb{C}) & = & K_0(\mathbb{C}) \end{array}$$

it is a simple verification to show that $[p] - [\ell_k] \in K_0(I)$ and so $[p] - [\ell_k] \in \ker(\pi_*)$. $\square$

Remark 2.4.1. We will often use short exact sequences where I is an ideal of A and $B = A/I$.

Now, in order to show the half-exactness of the other functors K_*, thanks to proposition 2.3.3, it suffices to discuss the exactness of the suspension.

Proposition 2.4.2. Let
$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$
be a short exact sequence of C^*-algebras. Then the induced sequence
$$0 \longrightarrow \mathcal{S}I \longrightarrow \mathcal{S}A \longrightarrow \mathcal{S}B \longrightarrow 0$$
is exact. In other words, the functor $\mathcal{S}$ is exact.

Proof. Denote by ι the second arrow of the short exact sequence and by π the third one. It is easy to see that ι is injective and that $\operatorname{im}(\mathcal{S}\iota) = \ker(\mathcal{S}\pi)$. It remains to show that π in surjective. Let $f \in C_0(\mathbb{R})$ and $b \in B$. By surjectivity of π, $b = \pi(a)$ for some $a \in A$. Then $fb = f\pi(a) = \mathcal{S}\pi(fa)$. So $\operatorname{span}\{fb,\ f \in C_0(\mathbb{R}),\ b \in B\} \subset \operatorname{im}(\mathcal{S}\pi)$. Hence, by lemma 1.3.1, $\mathcal{S}B = \operatorname{im}(\mathcal{S}\pi)$. □

It follows from the last two propositions and the natural isomorphism $K_1 \cong K_{-1}$ the half-exactness of K_n for every $n \leq 1$.

Proposition 2.4.3. Let
$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$
be a short exact sequence of C^*-algebras. Then the induced sequence
$$K_n(I) \longrightarrow K_n(A) \longrightarrow K_n(B)$$
is exact for every integer $n \leq 1$.

Proposition 2.4.4. Let
$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[l, bend right, "s"'] & 0 \end{tikzcd}$$
be a split short exact sequence of C^*-algebras. Then the induced sequence
$$\begin{tikzcd} 0 \arrow[r] & K_n(I) \arrow[r] & K_n(A) \arrow[r] & K_n(B) \arrow[r] \arrow[l, bend right, "s_*"'] & 0 \end{tikzcd}$$
is split exact for every integer $n \leq 1$.

Proof. The functor $\mathcal{S}$ is exact and it is easy to see that it is also split exact. Then it suffices to prove the statement for K_0. Denote by ι the second arrow of the short exact sequence and by π the third one. Since it splits, we have $\pi \circ s = id_B$, which induces $\pi_* \circ s_* = id_{K_0(B)}$. Thus the induced sequence splits, and π_* is surjective, which gives the exactness at $K_0(B)$. Furthermore, we already know from proposition 2.4.1 the exactness at $K_0(A)$. It just remains the exactness at $K_0(I)$, ie. injectivity of ι_*. Let $[p] - [\ell_k] \in \ker(\iota_*)$, where $p \in \mathcal{P}_n(I^+)$ and $k \leq n$. Then $[\iota^+(p)] = [\ell_k]$. Hence, in virtue of proposition 1.2.1, there is a unitary $u \in \mathcal{U}_n(A^+)$ such that $u\iota^+(p)u^* = \ell_k$. Set $v = s \circ \pi^+(u^*)u \in \mathcal{U}_n(A^+)$. Then $\pi^+(v - 1_n) = 0$, and so $v - 1_n \in \ker(\pi^+) = \operatorname{im}(\iota^+)$. It follows that there exists $w \in \mathcal{M}_n(I^+)$ such that $v = \iota^+(w)$. Note that w is unitary since ι^+ is injective. Then, direct computation gives us $\iota^+(wpw^*) = \ell_k = \iota^+(\ell_k)$. Hence $wpw^* = \ell_k$, ie. $p \sim_u \ell_k$ and so $[p] = [\ell_k]$. □

Now we can quickly prove that the functors K_* preserves direct sums.

Proposition 2.4.5. Let A and B two C^*-algebras. Then $K_n(A \oplus B) \cong K_n(A) \oplus K_n(B)$ for every $n \leq 1$.

Proof. It is simple to show that $\mathcal{S}$ preserves direct sums. Then it suffices to show the statement for K_0. We have the split short exact sequences
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A \oplus B \arrow[r] & B \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$
and
$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & A \oplus B \arrow[r] & A \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$
where the maps are obvious. By the previous proposition, they induce on K-theory the short exact sequences
$$0 \longrightarrow K_0(A) \longrightarrow K_0(A \oplus B) \longrightarrow K_0(B) \longrightarrow 0$$
and
$$0 \longrightarrow K_0(B) \longrightarrow K_0(A \oplus B) \longrightarrow K_0(A) \longrightarrow 0$$
Then it is easy to see that the following short sequence is exact, which gives us the conclusion.
$$0 \longrightarrow K_0(A) \oplus K_0(B) \longrightarrow K_0(A \oplus B) \longrightarrow K_0(A) \oplus K_0(B) \longrightarrow 0$$
□

2.5 Stability

In this subsection, we will prove another important property of K-theory : stability by tensoring by the C^*-algebra of compact operators on a Hilbert space, ie. $K_0(A \otimes \mathcal{K}(H)) \cong K_0(A)$. Remember that $\otimes$ denotes the minimal tensor product. Stability will allow us to prove Bott periodicity theorem later on. Before, we need some basic facts about direct limits of abelian groups : definition and two basic properties, which will not prove here. However one can find more about more general direct limits in [Wei94] or a simple proof of the universal property in [Mur90].
Consider the following sequence of homomorphisms of abelian groups :

$$G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-1}} G_n \xrightarrow{\varphi_n} \cdots$$

Set $G' = \left\{ (x_n)_{n\geq 1} \in \prod_{n\geq 1} G_n \;\middle|\; \exists N \geq 1, \forall n \geq N, x_{n+1} = \varphi_n(x_n) \right\}$. Then G' is a subgroup of $\prod_{n\geq 1} G_n$. The quotient $G'/\bigoplus_{n\geq 1} G_n$ is called the direct limit of the sequence above, denoted by $G = \varinjlim G_n$ when the homomorphisms are clear. Furthermore we define the natural homomorphism $\iota_n : G_n \to G$ by denoting by $\iota_n(x_n)$, for every $x_n \in G_n$, the image in the quotient of $(0_1, 0_2, \cdots, 0_{n-1}, x_n, 0_{n+1}, \cdots) \in G'$. The direct limit G has the following universal property.

Proposition 2.5.1 (universal property)**.** Let G' be an abelian group and, for each $n \geq 1$, let $\rho_n : G_n \to G'$ be a homomorphism such that the diagram

$$\begin{tikzcd} G_n \arrow[r, "\varphi_n"] \arrow[dr, "\rho_n"'] & G_{n+1} \arrow[d, "\rho_{n+1}"] \\ & G' \end{tikzcd}$$

commutes. Then there is a unique homomorphism $\rho : G \to G'$ such that for each $n \geq 1$ the diagram

$$\begin{tikzcd} G_n \arrow[r, "\iota_n"] \arrow[dr, "\rho_n"'] & G \arrow[d, "\rho"] \\ & G' \end{tikzcd}$$

commutes.

Proposition 2.5.2. Direct limit $\varinjlim$ is an exact functor from **Ab** to **Ab**.

Now let us study the behavior of K-theory on certain sequences of C^*-algebras, through the following proposition and corollary.

Proposition 2.5.3. Let $A_1, A_2, A_3, \cdots$ and A be unital C^*-algebras such that $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A$ and $\bigcup_{n\geq 1} A_n$ is dense in A. Then the induced map $\varinjlim K_0(A_n) \to K_0(A)$ is an isomorphism.

Proof. First let us prove injectivity. Let $p \in \mathcal{P}_k(A)$ (for some $k \geq 1$). Then, since $\bigcup_{n\geq 1} A_n$ is dense in A, there is $a \in A_m$ for some $m \geq 1$, such that $\|a - p\| \leq \frac{3}{190}$. Easy computations give us $\|a\| < 2$ and, knowing that $\|p\| = \max |\mathrm{sp}(p)| \leq 1$,

$$\left\|a^2 - a\right\| \leq \|a + p\| \, \|a - p\| + \|p - a\| + \|ap - aa^*\| + \|aa^* - pa^*\| + \|pa^* - pa\| < \frac{3}{19}$$

Hence $b = (a + a^*)/2$ is a self-adjoint (and so normal) element of $\mathcal{M}_k(A_m)$ such that $\|b - a\| < \frac{3}{190}$ and

$$\left\|b^2 - b\right\| \leq \|b + a\| \, \|b - a\| + \left\|a^2 - ab\right\| + \left\|ba - a^2\right\| + \left\|a^2 - a\right\| + \|a - b\| < \frac{3}{10}$$

Set $f(\lambda) = \lambda^2 - \lambda$. It defines a continuous function on $\mathbb{R}$. We deduce from the study of f that $-3/10 < f(\lambda) < 3/10$ implies $\lambda \in \left]-1/3\,, 1/3\right[\cup \left]2/3\,, 4/3\right[$. So, since $\left\|b^2 - b\right\| = \max\{\left|\lambda^2 - \lambda\right|, \lambda \in \mathrm{sp}(b)\}$ (using continuous functional calculus, see proposition 1.1.1), we get

$$\mathrm{sp}(b) \subset \left]-\frac{1}{3}, \frac{1}{3}\right[\cup \left]\frac{2}{3}, \frac{4}{3}\right[$$

Let χ be the characteristic function of the interval $[2/3\,, 4/3]$ and $q = \chi(b)$. Then $\chi(b)^* = \bar{\chi}(b) = \chi(b)$ and $\chi(b)^2 = \chi(b)$, ie. $q \in \mathcal{P}_k(A_m)$. Furthermore, by studying the fonction $\lambda \mapsto \chi(\lambda) - \lambda$ on $\mathrm{sp}(b)$, we get

$$\|q - p\| \leq \|\chi(b) - b\| + \frac{1}{2}\|a^* - a\| + \|a - p\| < \frac{1}{3} + \frac{3}{190} + \frac{3}{190} < 1$$

Hence, by considering the first part of the proof of proposition 1.2.2, replacing p_t by the constant path q, we get $u_t^* p u_t = q$ where $u_t \in \mathcal{U}_k(A_m)$ is a constant path of unitaries. So $p \sim_u q$ in $\mathcal{P}_k(A)$ and $[p] = [q]$ in $K_0(A)$. Thus $[p]$ is the image of the map $K_0(A_m) \to K_0(A)$, and so in the image of the induced map $\varinjlim K_0(A_n) \to K_0(A)$, considering the universal property of direct limits. This shows surjectivity.
For injectivity, let $p_0, p_1 \in \mathcal{P}_k(A_m)$ (for some $k, m \geq 1$) such that $p_0 \sim_h p_1$ in $\mathcal{P}_k(A)$. Then there is a path of projection p_t in $\mathcal{M}_k(A)$ from p_0 to p_1. By using compactness of $[0\,,1]$ and by considering an appropriate partition $0 = t_0 < t_1 < \cdots t_N = 1$ of $[0\,,1]$, one can easily show that $\bigcup_{n\geq 1} C([0,1], A_n)$ is dense in $C([0,1], A)$ and so that $C([0,1], A_1) \subset C([0,1], A_2) \subset \cdots \subset C([0,1], A)$ fulfill the condition of the theorem we are proving. So we can apply the same reasoning as in the first part of this proof to find a projection q_t of $\mathcal{M}_k(C([0,1], A_{m'})) \cong C([0,1], \mathcal{M}_k(A_{m'}))$ for some $m' \geq m$, such that $q_0 = p_0$ and $q_1 = p_1$. Thus $p_0 \sim_h p_1$ in $\mathcal{P}_k(A_{m'})$ and so $[p_0] = [p_0]$ in $K_0(A_{m'})$. $\square$

This result extends to non-unital C^*-algebras.

Corollary 2.5.1. Let $A_1, A_2, A_3, \cdots$ and A be C^*-algebras such that $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A$ and $\bigcup_{n\geq 1} A_n$ is dense in A. Then $\varinjlim K_0(A_n) \cong K_0(A)$.

Proof. Remember that the short exact sequence

$$0 \longrightarrow A_n \longrightarrow A_n^+ \longrightarrow \mathbb{C} \longrightarrow 0$$

splits and so induces a short exact sequence on K-theory :

$$0 \longrightarrow K_0(A_n) \longrightarrow K_0(A_n^+) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Then, by proposition 2.5.2 (exactness of direct limits) and using the previous proposition with A^+ since $\varinjlim A_n^+ = A^+$, we get the short exact sequence

$$0 \longrightarrow \varinjlim K_0(A_n) \longrightarrow K_0(A^+) \longrightarrow \mathbb{Z} \longrightarrow 0$$

Thus, by definition of K_0 for non-unital C^*-algebras, we have $\varinjlim K_0(A_n) \cong \ker(K_0(A^+) \to \mathbb{Z}) = K_0(A)$. $\square$

The last step before concluding on stability of K-theory is the following proposition, first for unital C^*-algebras and then for non-unital C^*-algebras as previously.

Proposition 2.5.4. Let A be a unital C^*-algebra and $n \geq 1$. Then $K_0(\mathcal{M}_n(A)) \cong K_0(A)$.

Proof. For every $k \geq 1$, consider the map

$$\begin{array}{rccc} m_k^n : & \mathcal{M}_k(A) & \longrightarrow & \mathcal{M}_{kn}(A) \\ & a & \longmapsto & a \oplus 0_{k(n-1)} \end{array}$$

Then we have the commutative diagram

$$\begin{tikzcd} \mathcal{P}(A) \arrow[r] \arrow[d, "m_1^n"] & \mathcal{P}_2(A) \arrow[r] \arrow[d, "m_2^n"] & \cdots \arrow[r] & \mathcal{P}_k(A) \arrow[r] \arrow[d, "m_k^n"] & \cdots \\ \mathcal{P}_n(A) \arrow[r] & \mathcal{P}_{2n}(A) \arrow[r] & \cdots \arrow[r] & \mathcal{P}_{kn}(A) \arrow[r] & \cdots \end{tikzcd}$$

So, by considering the universal property of direct limits of topological spaces, we get an induced map

$$\begin{array}{rccc} m^n : & \lim_k \mathcal{P}_k(A) & \longrightarrow & \lim_k \mathcal{P}_{kn}(A) \\ & [p] & \longmapsto & [p \oplus 0_{k(n-1)}] \end{array}$$

whose the inverse is

$$\begin{array}{rccc} m^n : & \lim_k \mathcal{P}_{kn}(A) & \longrightarrow & \lim_k \mathcal{P}_k(A) \\ & [p] & \longmapsto & [p] \end{array}$$

Therfore m^n is a bijection and so it induces a bijection $m^n : V(A) \to V(\mathcal{M}_n(A))$ between the monoids $V(A)$ and $V(\mathcal{M}_n(A))$. Furthermore, given $p \in \mathcal{P}_k(A)$ and $q \in \mathcal{P}_l(A)$, we have

$$m_{k+l}^n(p \oplus q) = p \oplus q \oplus 0_{(k+l)(n-1)} \sim_h p \oplus 0_{k(n-1)} \oplus q \oplus 0_{l(n-1)} = m_k^n(p) \oplus m_l^n(q)$$

by proposition 1.2.1. Thus m^n is an isomorphism of abelian monoids. So it induces an isomorphism between the groups $K_0(A)$ and $K_0(\mathcal{M}_n(A))$. $\square$

Corollary 2.5.2. Let A be a C^*-algebra and $n \geq 1$. Then $K_0(\mathcal{M}_n(A)) \cong K_0(A)$.

Remark 2.5.1. In fact, one can show that we have a natural isomorphism between the functors K_0 and $K_0 \circ \mathcal{M}_n$ for all $n \geq 1$. This property is called Morita invariance.

Finally we can deduce stability of K-theory. For an Hilbert space H, $\mathcal{K}(H)$ denotes the algebra of compact operators on H.

Proposition 2.5.5. Let A be a unital C^*-algebra. Then $\mathbb{K}_0(A \otimes \mathcal{K}(H)) \cong K_0(A)$ for any separable Hilbert space H.

Proof. If H is finite-dimensional, then this proposition is just the previous corollary. Now suppose that H is infinite-dimensional. Let $(h_n)_{n\geq 1}$ be a Hilbert basis of H. Set $H_n = \overline{\operatorname{span}(h_i)_{1\leq i\leq n}}$ for every $n \geq 1$. Then, since finite rank operators are dense in $\mathcal{K}(H)$, $\bigcup_{n\geq 1} A \otimes \mathcal{B}(H_n)$ is dense in $A \otimes \mathcal{K}(H)$. Besides, $\mathcal{B}(H_n) \cong \mathcal{M}_n(\mathbb{C})$, and $A \otimes \mathcal{M}_n(\mathbb{C}) \cong \mathcal{M}_n(A)$ by the homomorphism which maps $a \otimes M$ to the matrix $M(a)$ whose coefficients are those of M times a. Thus, by the previous two corollaries, $K_0(A) \cong \varinjlim K_0(A \otimes \mathcal{B}(H_n)) \cong K_0(A \otimes \mathcal{K}(H))$ ☐

The following corollary comes from the fact that $\mathcal{S}A \cong C_0(\mathbb{R}) \otimes A$.

Corollary 2.5.3. Let A be a C^*-algebra. Then $\mathbb{K}_n(A \otimes \mathcal{K}(H)) \cong K_n(A)$ for any separable Hilbert space H and $n \leq 1$.

Remark 2.5.2. We often write $K_0(A \otimes \mathcal{K}) \cong K_0(A)$, where $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$.

3 Bott periodicity

One of the most important results of K-theory is Bott periodicity theorem, which will give us an isomorphism between $K_{-2}(A)$ and $K_0(A)$ and gives rise to a 6-term exact sequence from the half infinite exact sequence. This 6-term exact sequence follows from the index map and is a very powerful tool to compute the K-theory of some C^*-algebras. In fact, the purpose of this section is to prove Bott periodicity theorem and then the 6-term exact sequence as a final result. The proof we will study here is a proof of Joachim Cuntz that we can read in his article [Cun84] or in [NdK16] and [Mur90]. It mainly consists of showing K-contractibility of a certain C^*-subalgebra of the Toeplitz algebra. This proof has the advantage of not using directly the definitions of K_0 and K_1 and so of being very general by relying only on homotopy invariance, half-exactness and stability of the functor. The following first subsection is based on [Bla86] with lemmas coming from [NdK16].

3.1 Half infinite exact sequence

First, let us construct the half infinite exact sequence and introduce the index map, which links K_1 to K_0. Let A be a C^*-algebra and I an ideal of A. Then we have the short exact sequence of C^*-algebras

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0$$

Then, recall that it induces the following exact sequences on K-theory :

$$K_1(I) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\pi_*} K_1(A/I) \quad \text{and} \quad K_0(I) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I)$$

One wants to define a map from $K_1(A/I)$ to $K_0(I)$ such that the sequence obtained from the concatenation of these two sequences is exact. Note that $A^+/I \cong (A/I)^+$ via the factorization of the surjective homomorphism $a+\lambda 1 \to \pi(a)+\lambda 1$, and denote by $\pi_+ : A^+ \to A^+/I$ the quotient map. Let $[u] \in K_1(A/I)$, where $u \in \mathcal{U}_n(A^+/I)$. Then, in virtue of lemma 2.4.1, there is a lift $z \in \mathcal{U}_{2n}(A^+)$ of $u \oplus u^*$: $\pi_+(z) = u \oplus u^*$. Set

$$\partial([u]) = [z\ell_n z^*] - [\ell_n]$$

Before proving that this expression defines the wanted map, we need the following lemma.

Lemma 3.1.1. Given a C^*-algebra A and an ideal $I \subset A$, we have

$$C([0\,,1], A) \Big/ C([0\,,1], I) \cong C([0\,,1], A/I)$$

Proof. Consider the map $\Phi : C([0\,,1], A) \to C([0\,,1], A/I)$ defined by $\varphi(f) = \pi \circ f$, where $\pi : A \to A/I$ is the quotient map. Then Φ is a homomorphism of C^*-algebras and is valued in $C([0\,,1], A/I)$ as well (it follows directly from the definition of the norm on A/I). A simple verification shows that $\ker \Phi = C([0\,,1], I)$. For surjectivity, note that

$$\{f\pi(a),\ f \in C([0\,,1]),\ a \in A\} \subset \mathrm{im}\Phi \subset C([0\,,1], A/I)$$

So, since imΦ is closed (see remark 1.1.3) and the first set on the left is dense in $C([0\,,1], A/I)$ by lemma 1.3.1, Φ is surjective. Thus we get an isomorphism by factorization. $\square$

Proposition 3.1.1. ∂ is a homomorphism from $K_1(A/I)$ to $K_0(I)$, called the index map.

Proof. Let us show that ∂ is well defined. Let $[u] \in K_0(A)$ and $z \in \mathcal{U}_{2n}(A^+)$ as above. First, $\partial([u]) \in K_0(I)$. Indeed, a direct computation tells us that $\pi_+(z\ell_n z^*) = \ell_n$. So $z\ell_n z^* - \ell_n \in \mathcal{M}_{2n}(I) \subset \mathcal{M}_{2n}(I^+)$. Since $\ell_n \in \mathcal{M}_{2n}(I^+)$, we get that $z\ell_n z^* \in \mathcal{M}_{2n}(I^+)$, and it is easily seen to be a projection. Furthermore, since $z\ell_n z^* \equiv \ell_n \mod I$, $\partial([u]) = [z\ell_n z^*] - [\ell_n] \in \ker(K_0(I^+) \to K_0(\mathbb{C})) = K_0(I)$. Now we prove that $\partial([u])$ does not depend on the lift z. Let $z' \in \mathcal{U}_{2n}(A^+)$ be another lift of $u \oplus u^*$. Then $z'z^* \in \mathcal{U}_{2n}(A^+)$ and $\pi_+(z'z^*) = 1_{2n}$. Hence $z'z^* - 1_{2n} \in \mathcal{M}_{2n}(I) \subset \mathcal{M}_{2n}(I^+)$. That is why $z'z^* \in \mathcal{U}_{2n}(I^+)$. Moreover this unitary conjugates $z\ell_n z^*$ to $z'\ell_n z'^*$: $z\ell_n z^* \sim_u z'\ell_n z'^*$ in $\mathcal{P}_{2n}(I^+)$. So $[z'\ell_n z'^*] = [z\ell_n z^*]$, which shows that $\partial([u])$ does not depend on the lift. It remains to prove that it does not depends on the representing element of the class. Let $v \in \mathcal{U}_n(A^+/I)$ such that $[u] = [v]$. Then there is a path of unitaries y_t from u to v. Hence, by lemma 3.1.1, this path defines an element of $C([0\,,1], A^+)/C([0\,,1], I)$, and so, by lemma 2.4.1, there is a lift $x_t \in \mathcal{U}_{2n}(C([0\,,1], A^+))$ of $y_t \oplus y_t^*$. Then $\pi_+(x_0) \in \mathcal{U}_{2n}(A^+)$ is a lift of $u \oplus u^*$, $\pi_+(x_1)$ is a lift of $v \oplus v^*$, and $\pi_+(x_t)\ell_n\pi_+(x_t)^*$ is a path of projections. Thus

$$\partial([u]) = [\pi_+(x_0)\ell_n\pi_+(x_0)^*] - [\ell_n] = [\pi_+(x_1)\ell_n\pi_+(x_1)^*] - [\ell_n] = \partial([v])$$

Finally, ∂ is easily seen to be a homomorphism. $\square$

In order to show exactness, we need the following lemma on path lifting of unitaries. Its proof involves holomorphic functional calculus, which we do not describe here. However one can find an introduction to this theory in [DS88].

Lemma 3.1.2. Let u_t be a path in $\mathcal{U}(A/I)$ and $U \in \mathcal{U}(A)$ such that $\pi(U) = u_0$, where A is a unital C^*-algebra and $I \subset A$ an ideal. Then there exists a path U_t in $\mathcal{U}(A)$ such that $\pi(U_t) = u_t$ for all $t \in [0\,,1]$ and $U_0 = U$.

Proof. First suppose that $\|u_0^* u_t - 1\| < 1$ for all $t \in [0\,,1]$. Then, since ln is a holomorphic function on a neighborhood of $\bigcup_{t\in[0,1]} \mathrm{sp}(u_0^* u_t) \subset \{z \in \mathbb{C} \mid \Re(z) > 0\}$, $w_t = \ln(u_0^* u_t)$ defines a path in A/I. Hence, by lemma 3.1.1, there is a lift W_t in $C([0\,,1], A)$ of the path w_t. We can choose W_t starting at 0, by replacing W_t by $W_t - W_0$ if necessary, since $w_0 = 0$. Define $Z_t = Ue^{W_t}$. Then Z_t is a path of invertible in A. So, by considering the polar decomposition of Z_t for all $t \in [0\,,1]$ (see proposition 1.1.3), we find that $U_t = Z_t\,|Z_t|^{-1}$ is a path of unitaries in A. Furthermore $\pi(U_t) = u_0 u_0^* u_t (u_t^* u_0 u_0^* u_t)^{-1/2} = u_t$ and $U_0 = U$.

Now consider the general case. Then, since u_t is continuous on the compact $[0\,,1]$, there exists a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\forall t \in [t_i\,, t_{i+1}]$, $\|u_{t_i} - u_t\| < C^{-1}$, where $C > 0$ is such that $\sup_{t\in[0,1]} \|u_t\| \leq C$. So $\left\|u_{t_i}^* u_t - 1\right\| < 1$. Hence, we can apply the first point iteratively with $U_{t_i}^{i-1}$ in the role of U, to get a path U_t^i on each interval $[t_i\,, t_{i+1}]$. Thus, by gluing the paths U_t^i, we obtain the desired path of unitaries. ☐

Proposition 3.1.2. The sequence

$$K_1(I) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\pi_*} K_1(A/I) \xrightarrow{\partial} K_0(I) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/I)$$

is exact.

Proof. First, let us prove the exactness at $K_1(A/I)$. Let $u \in \mathcal{U}_n(A^+)$ for some $n \geq 1$. Then $\pi_+(u)$ is unitary in $\mathcal{M}_n(A^+/I)$, and $u \oplus u^*$ is a lift of $\pi_+(u) \oplus \pi_+(u)^*$. Hence, since $(u \oplus u^*)\ell_n(u^* \oplus u) = 0$, we have $\partial([u]) = [(u \oplus u^*)\ell_n(u \oplus u^*)^*] - [\ell_n] = 0$, which shows that $\mathrm{im}(\pi_*) \subset \ker\partial$. Now, let $u \in \mathcal{U}_n(A^+/I)$ such that $\partial([u]) = 0$, ie. $[z\ell_n z^*] = [\ell_n]$ for some lift $z \in \mathcal{U}_{2n}(A^+)$ of $u \oplus u^*$. Then there exists $w \in \mathcal{U}_{2n}(I^+)$ which conjugates $z\ell_n z^*$ to ℓ_n. We have $w \equiv \lambda \mod I$, ie. $\pi_+(w) = \lambda$, where $\lambda \in \mathcal{U}_{2n}(\mathbb{C})$. Set $x = \lambda^* wz$. Then, since $\lambda^* w \equiv\in \mathcal{U}_{2n}(I^+)$, $x \in \mathcal{U}_{2n}(I^+)$. Note that x commutes with ℓ_n. Hence x is must be of the form

$$x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where $a, b \in \mathcal{U}_n(I^+)$. Furthermore, a direct computation shows that $\pi_+(x) = u \oplus u^*$. So $\pi_+(a) = u$, ie. a is a lift of u, and so $[u] = \pi_*([a]) \in \mathrm{im}(\pi_*)$. This finishes to prove the exactness at $K_1(A/I)$.

For exactness at $K_0(I)$, we have $\mathrm{im}\partial \subset \ker(\iota_*)$. Indeed, given $u \in \mathcal{U}_n(A^+/I)$ and $z \in \mathcal{U}_{2n}(A^+)$ a lift of $u \oplus u^*$, $\iota_*(\partial([u])) = [\iota^+(z\ell_n z^*)] - [\ell_n] = 0$ since $\iota^+(z\ell_n z^*) = z\ell_n z^* \sim_u \ell_n$ in $\mathcal{P}_{2n}(A^+)$. Conversely, let $x \in \ker\iota_* \subset K_0(I)$. This element can be written as a difference $[p] - [\ell_n]$ where $k \geq 1$ and $p \in \mathcal{P}_{kn}(I^+)$ (see corollary 2.2.1), such that $\iota_*([p] - [\ell_n]) = 0$, ie. $[p] = [\ell_n]$ in $K_0(A^+)$. Then there exists $w \in \mathcal{U}_{kn}(A^+)$ such that $p = w\ell_n w^*$. Furthermore, since $[p] - [\ell_n] \in \ker(K_0(A^+) \to K_0(\mathbb{C}))$ and so the projection of p on $\mathbb{C}$ is unitarily equivalent to ℓ_n in $\mathcal{P}_{kn}(\mathbb{C})$, we may assume that this projection of p is ℓ_n. Hence we have $\pi_+(p) = \ell_n$ and so $\pi_+(w)$ commutes with ℓ_n. It follows that $\pi_+(w)$ is of the form

$$\pi_+(w) = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

where $u_1 \in \mathcal{U}_n(A^+/I)$ and $u_2 \in \mathcal{U}_{(k-1)n}(A^+/I)$. Then, in $\mathcal{U}_{kn}(A^+/I)$, we have

$$\begin{pmatrix} u_1^* & 0 \\ 0 & 1_{(k-1)n} \end{pmatrix} \begin{pmatrix} u_2^* & 0 \\ 0 & 1_n \end{pmatrix} \sim_h \begin{pmatrix} u_1^* & 0 \\ 0 & 1_{(k-1)n} \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & u_2^* \end{pmatrix} = \pi_+(w^*)$$

ie. there is a path of unitaries from $\pi_+(w)$ to $(u_1^* \oplus 1_{(k-1)n})(u_2^* \oplus 1_n)$. So, by lemma 3.1.2, this last unitary matrix has a unitary lift $v \in \mathcal{U}_{kn}(A^+)$. Set $z = (1_n \oplus v)(w \oplus 1_n)$. Hence $\pi_+(z) = u_1 \oplus u_1^* \oplus 1_n$ and

$$z\ell_n z^* = \begin{pmatrix} 1_n & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0_n \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & v^* \end{pmatrix} \sim_u \begin{pmatrix} p & 0 \\ 0 & 0_n \end{pmatrix}$$

Thus $[p] - [\ell_n] = [z\ell_n z^*] - [\ell_n] = \partial([u_1]) \in \mathrm{im}(\partial)$. And so $\ker\iota_* \subset \mathrm{im}(\partial)$. ☐

From this we are able to deduce the following half-infinite exact sequence.

Proposition 3.1.3. If A is a C^*-algebra and I is an ideal of A, then there exists a half-infinite exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & K_{-n}(I) \arrow[r] & K_{-n}(A) \arrow[r] & K_{-n}(A/I) \arrow[r] & \cdots \arrow[r] & K_{-2}(I) \arrow[r] & K_{-2}(A) \arrow[r] & K_{-2}(A/I) \arrow[d] \\
 & & K_0(A/I) & K_0(A) \arrow[l] & K_0(I) \arrow[l] & K_{-1}(A/I) \arrow[l] & K_{-1}(A) \arrow[l] & K_{-1}(I) \arrow[l]
\end{tikzcd}$$

Proof. First note that it follows from exactness of $\mathcal{S}$ (see proposition 2.4.2) that $\mathcal{S}(A/I) \cong \mathcal{S}A/\mathcal{S}I$. Then, using proposition 2.3.3 and applying the previous proposition to A and its suspension $\mathcal{S}A$, we get the following two exact sequences

$$K_{-1}(\mathcal{S}I) \to K_{-1}(\mathcal{S}A) \to K_{-1}(\mathcal{S}(A/I)) \to K_0(\mathcal{S}I) \to K_0(\mathcal{S}A) \to K_0(\mathcal{S}(A/I))$$

and

$$K_{-1}(I) \to K_{-1}(A) \to K_{-1}(A/I) \to K_0(I) \to K_0(A) \to K_0(A/I)$$

Hence, by definition of K_{-1} and because the last three arrows of the first sequence are the same as the first three arrows of the second sequence, we obtain the exact sequence

$$\begin{tikzcd}
K_{-2}(I) \arrow[r] & K_{-2}(A) \arrow[r] & K_{-2}(A/I) \arrow[r] & K_{-1}(I) \arrow[r] & K_{-1}(A) \arrow[r] & K_{-1}(A/I) \arrow[d] \\
 & & & K_0(A/I) & K_0(A) \arrow[l] & K_0(I) \arrow[l]
\end{tikzcd}$$

Remembering that $K_{-2}(A) = K_0(\mathcal{S}^2 A)$ and by iteration on the order of the suspension, we extend the sequence to the left to get the half-infinite exact sequence. $\square$

3.2 The Toeplitz algebra

Now let us define the Toeplitz algebra $\mathcal{T}_0$, whose the extension $0 \to \mathcal{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0$ will be useful in the proof of the Bott periodicity theorem. We will see its main properties and especially different ways of representing it.
Denote the unit circle S^1 of $\mathbb{C}$ by $\mathbb{T}$ and consider the space $L^2(\mathbb{T})$ with the scalar product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,\mathrm{d}t$$

It turns $L^2(\mathbb{T})$ into an Hilbert space whose $(e_n)_{n\in\mathbb{Z}}$ is a Hilbert basis, where

$$e_n : \begin{array}{ccc} \mathbb{T} & \longrightarrow & \mathbb{T} \\ z & \longmapsto & z^n \end{array}$$

Then consider the subspace

$$H^2 = \left\{ f \in L^2(\mathbb{T}) \;\middle|\; f = \sum_{n\geq 0} f_n e_n,\ (f_n)_{n\geq 0} \in \ell^2(\mathbb{N}) \right\}$$

Let $P : L^2(\mathbb{T}) \to H^2$ be the orthogonal projection onto H^2.

Definition 3.2.1. For every $f \in C(\mathbb{T})$, we call the Toeplitz operator with symbol f the bounded operator $T_f : H^2 \to H^2$ defined by

$$T_f(g) = P(fg)$$

Remark 3.2.1. For every $f \in C(\mathbb{T})$, $T_f^* = T_{\bar{f}}$.

Definition 3.2.2. The Toeplitz algebra is defined as the closure of $\{T_f,\ f \in C(\mathbb{T})\}$ in $\mathcal{B}(H^2)$

Remark 3.2.2. Denote by S the right shift T_{e_1} of H^2 with respect to the basis $(e_n)_{n\in\mathbb{N}}$ as well as the right shift of $\ell^2(\mathbb{N})$. SS^* is easily seen to be a (orthogonal) projection onto $\ell^2(\mathbb{N}^*)$. Then $S^n(1-SS^*)S^m$ is the rank 1 operator sending e_m to e_n. This implies that $\mathcal{K}$, the ideal of compact operators, is included in $\mathcal{T}$. Furthermore, for all $n, m \in \mathbb{N}$, $T_{e_n}T_{e_m} - T_{e_{n+m}}$ is easily seen to be a compact operator. Thus, by the Stone-Weierstrass theorem and since an easy computation shows that $\forall f, g \in C(\mathbb{T})$, $\|T_fT_g - T_{fg}\| \leq \|f\|_\infty \|g\|_\infty$, $T_fT_g - T_{fg}$ is compact for every $f, g \in C(\mathbb{T})$.

As a straightforward consequence of this remark, we have the following characterization of the Topeplitz algebra $\mathcal{T}$.

Proposition 3.2.1. The Topeplitz algebra $\mathcal{T}$ is isomorphic to the C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$ generated by S.

In fact, we have the following theorem by L.A. Coburn proved in [Mur90] and [Cob67].

Theorem 3.2.1. The Topeplitz algebra $\mathcal{T} \cong C^*(S)$ is the universal unital C^*-algebra generated by an isometry : for any unital C^*-algebra A and any isometry $w \in A$, there is a unique unital homomorphism $\mathcal{T} \to A$ which sends S to w.

Let us conclude this subsection on the Toeplitz algebra with the following important proposition.

Proposition 3.2.2. We have the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to C(\mathbb{T}) \to 0$$

Proof. Let $\pi : \mathcal{T} \to \mathcal{T}/\mathcal{K}$ the quotient map. Since S generates $\mathcal{T}$, $s = \pi(S)$ generates $\mathcal{T}/\mathcal{K}$. Furthermore, since $1 - SS^*$ and $SS^* - S^*S$ both lie in $\mathcal{K}$, s is unitary. Then, by the continuous functional calculus, $\mathcal{T}/\mathcal{K}$ is isomorphic to $C(\mathrm{sp}(s))$. Besides, $\mathrm{sp}(s) \subset \mathbb{T}$. For every $\lambda \in \mathbb{T}$, as a consequence of the previous theorem, there is a homomorphism from $\mathcal{T}/\mathcal{K}$ to $\mathbb{C}$ which maps s to λ. So $\mathrm{sp}(s) = \mathbb{T}$ and $\mathcal{T}/\mathcal{K} \cong C(\mathbb{T})$. The conclusion comes the standard short exact sequence

$$0 \to \mathcal{K} \to \mathcal{T} \to \mathcal{T}/\mathcal{K} \to 0$$

□

3.3 K-contractibility of $\mathcal{T}_0$

First set $\mathcal{T}_0$ as the ideal of the Toeplitz algebra $\mathcal{T}$ generated by $S - 1$. This section aims at proving the K-contractibility of $\mathcal{T}_0$: $K_0(\mathcal{T}_0 \otimes A) = 0$ for any C^*-algebra A. This is the main step to prove the Bott periodicity theorem. Let A be any C^*-algebra. We have the following diagram with horizontal and vertical short exact sequences :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, equal] & \mathcal{T}_0 \arrow[r, "q"] \arrow[d] & C_0(\mathbb{R}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{T} \arrow[r, "q"] \arrow[d, "j"] & C(\mathbb{T}) \arrow[r] \arrow[d, "ev_1"] & 0 \\
 & 0 \arrow[r] & \mathbb{C} \arrow[r, equal] \arrow[d] & \mathbb{C} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $j : \mathcal{T} \to \mathbb{C}$ is the homomorphism which maps S to 1 and $q : \mathcal{T} \to C(\mathbb{T})$ the one which maps S to $e_1 : z \mapsto z$. The inclusion $\mathcal{K} \to \mathcal{T}_0$ comes from the fact that $1 - SS^* = -(S-1)S^* - (S-1)^*$ and remark 3.2.2, since $\mathcal{T}_0$ is an ideal. This diagram is easily seen to be commutative apart from the map $\mathcal{T}_0 \to C_0(\mathbb{T})$. In fact, this map is defined in order to make the whole diagram commutative by noting that, since $ev_1 \circ q = j = 0$ on $\mathcal{T}_0$, the image of $\mathcal{T}_0$ through q lies in $C_0(\mathbb{T})$. Now, since $\mathbb{C}$, $C_0(\mathbb{T})$ and $C(\mathbb{T})$ are commutative and so nuclear (see proposition 1.3.2), the short sequences of the diagram above remain exact after tensoring by A by proposition 1.3.5. Furthermore the digram still commutes. Remember that $\mathbb{C} \otimes A \cong A$ and $C_0(\mathbb{R}) \otimes A \cong \mathcal{S}A$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K}\otimes A \arrow[r] \arrow[d, equal] & \mathcal{T}_0\otimes A \arrow[r, "q\otimes id"] \arrow[d] & C_0(\mathbb{R})\otimes A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K}\otimes A \arrow[r] \arrow[d] & \mathcal{T}\otimes A \arrow[r, "q\otimes id"] \arrow[d, "j\otimes id"] & C(\mathbb{T})\otimes A \arrow[r] \arrow[d, "ev_1\otimes id"] & 0 \\
 & 0 \arrow[r] & A \arrow[r, equal] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Let $\varepsilon : \mathcal{T} \to \mathbb{C}1_{\mathcal{T}}$ be the homomorphism given by $\forall x \in \mathcal{T},\ \varepsilon(x) = q(x)(1)1_{\mathcal{T}}$.

Lemma 3.3.1. For any C^*-algebra A, we have

$$K_0(\mathcal{T}_0 \otimes A) = \ker(K_0(\mathcal{T}\otimes A) \xrightarrow{(\varepsilon\otimes id)_*} K_0(\mathcal{T}_0\otimes A))$$

Proof. It follows easily from the definition of ε that $K_0(\mathcal{T}_0\otimes A) \subset \ker(\varepsilon\otimes id)_*$ since $\varepsilon(S) = 1$. Furthermore, the map $s \otimes id$, where $s : \mathbb{C} \hookrightarrow \mathcal{T}$, split the middle column. Hence, by proposition 2.4.4, $K_0(\mathcal{T}_0 \otimes A) = \ker(j \otimes id)_*$. Then, for every $x \in \ker(\varepsilon \otimes id)_*$, we have

$$x \in \ker(j\otimes id)_* \circ (\varepsilon\otimes id)_* = \ker(ev_1 \otimes id)_* \circ (q\otimes id)* = \ker(j\otimes id)_* = K_0(\mathcal{T}_0\otimes A)$$

$\square$

Lemma 3.3.2. Let A and B be unital C^*-algebras and let $f, g : A \to B$ two homomorphisms such that $fg = gf = 0$. Then $f + g$ defines a homomorphism and $(f + g)_* = f_* + g_*$.

Proof. It is easy to check that $f + g$ is a homomorphism. Let $[p] - [q] \in K_0(A)$, where $p, q \in \mathcal{M}_n(A)$ for some $n \geq 1$. Then, using proposition 2.2.2, we get

$$(f+g)_*([p]-[q]) = [f(p)+g(p)] - [f(q)+g(q)] = [f(p)]+[g(p)]-[f(q)]-[g(q)] = (f_*+g_*)([p]-[q])$$

$\square$

Then, to prove the K-contractibility of $\mathcal{T}_0$, we will show that $\ker(\varepsilon \otimes id)_* = 0$ by proving that the map $\varepsilon \otimes id$ is homotopic to the identity in order to apply the homotopy invariance of K_0 (see proposition 2.2.4). Consider the C^*-algebra $\widehat{\mathcal{T}} = \mathcal{K}\otimes\mathcal{T} + \mathcal{T}\otimes 1 \subset \mathcal{T}\otimes\mathcal{T}$. Then it easily seen to fit in the exact sequence

$$0 \to \mathcal{K}\otimes\mathcal{T} \to \widehat{\mathcal{T}} \xrightarrow{\widehat{q}} C(\mathbb{T}) \to 0$$

where the first map is the inclusion and $\widehat{q}$ is the composition of $q : \mathcal{T} \to C(\mathbb{T})$ and the multiplication in $\mathcal{T}$. Then define the C^*-subalgebra of $\widehat{\mathcal{T}} \oplus \mathcal{T}$:

$$\overline{\mathcal{T}} = \left\{(f,g) \in \widehat{\mathcal{T}}\oplus\mathcal{T} \;\middle|\; \widehat{q}(f) = q(g)\right\}$$

Then it fits in the short exact sequence

$$0 \to \mathcal{K}\otimes\mathcal{T} \to \overline{\mathcal{T}} \to \mathcal{T} \to 0$$

which splits by the map given by $x \in \mathcal{T} \mapsto (x \otimes 1, x)$. Now, let $\alpha_0, \alpha_1 : \mathcal{T} \to \overline{\mathcal{T}}$ be the maps given by $\alpha_0(S) = (p_0 \otimes S, 0)$ and $\alpha_1(S) = (p_0 \otimes 1, 0)$, where $p_0 = 1 - SS^* \in \mathcal{K}$ is the projection onto the first coordinate in $\ell^2(\mathbb{N})$. Note that $\mathrm{im}(\alpha_i) \subset (p_0 \otimes 1, 0)\overline{\mathcal{T}}(p_0 \otimes 1, 0)$ and so $\alpha_i(S)$ is an isometry, which shows that the maps are well defined by the universal property of $\mathcal{T}$. On the same way, define β by $\beta(S) = (S(1 - p_0) \otimes 1, S)$, which is well defined since $\mathrm{im}(\beta) \subset ((1 - p_0) \otimes 1, 1)\overline{\mathcal{T}}((1 - p_0) \otimes 1, 1)$. Note that $\alpha_i\beta = \beta\alpha_i = 0$ so that $\alpha_i + \beta$ are homomorphisms of C^*-algebras by the previous lemma.

Lemma 3.3.3. The homomorphisms $\alpha_0 + \beta$ and $\alpha_1 + \beta$ are homotopic.

Proof. Since S generates $\mathcal{T}$, it suffices to show that the isometries $\alpha_0(S)+\beta(S)$ and $\alpha_1(S)+\beta(S)$ are connected by a path of isometries s_t. Then $H_t(S)=s_t$ will defines a homotopy from $\alpha_0+\beta$ to $\alpha_1+\beta$. We have

$$\alpha_0(S)+\beta(S)=(p_0\otimes S+S(1-p_0)\otimes 1,S) \quad \text{and} \quad \alpha_1(S)+\beta(S)=((p_0+S(1-p_0))\otimes 1,S)$$

Set $u_0=S(1-p_0)S^*\otimes 1+p_0S^*\otimes S+Sp_0\otimes S^*+p_0\otimes p_0$. Then a simple verification shows that u_0 is a self-adjoint unitary of $\widehat{\mathcal{T}}$. Set also $u_1=(1+p_0(S^*-1)+p_1(S-1))\otimes 1$, which is also a self-adjoint unitary of $\widehat{\mathcal{T}}$, where $p_1\in\mathcal{K}$ is the projection onto the second coordinate in $\ell^2(\mathbb{N})$. Then we have

$$\alpha_0(S)+\beta(S)=(u_0(S\otimes 1),S) \quad \text{and} \quad \alpha_1(S)+\beta(S)=(u_1(S\otimes 1),S)$$

Since the unitaries u_0 and u_1 are self-adjoint, their spectrum is contained in $\{-1,1\}$. Hence $\exp(t\ln(u_i))$ are path of unitaries connecting u_0 and u_1 to 1. So it gives us a path u_t connecting u_0 to u_1. Furthermore, since $\widehat{q}(u_0)=\widehat{q}(u_1)=1$ (because $\mathcal{K}=\ker(q)$), $\forall t\in[0,1]$, $\widehat{q}(u_t)=1$. Then $s_t=(u_t(S\otimes 1),S)$ defines a path of isometries in $\overline{\mathcal{T}}$ from $\alpha_0(S)+\beta(S)$ to $\alpha_1(S)+\beta(S)$. □

Now we can conclude about the K-contractibility of $\mathcal{T}_0$.

Theorem 3.3.1. For every C^*-algebra A, $K_0(\mathcal{T}_0\otimes A)=0$.

Proof. By the previous lemma, we have two homotopic homomorphisms $\gamma_0=(\alpha_0+\beta)\otimes id$ and $\gamma_1=(\alpha_1+\beta)\otimes id$ from $\mathcal{T}\otimes A$ to $\overline{\mathcal{T}}\otimes A$. Hence, by proposition 2.2.4, they induce the same map on K-theory. So, by lemma 3.3.2,

$$(\alpha_0\otimes id)_*+(\beta\otimes id)_*=(\alpha_1\otimes id)_*+(\beta\otimes id)_*$$

and so $(\alpha_0\otimes id)_*=(\alpha_1\otimes id)_*$. Note that, for every $x\in\mathcal{T}$, $\alpha_0(x)=p_0\otimes x$ and $\alpha_1(x)=p_0\otimes\varepsilon$. Remember that the short exact sequence

$$0\to\mathcal{K}\otimes\mathcal{T}\to\overline{\mathcal{T}}\to\mathcal{T}\to 0$$

splits and so, denoting by i the inclusion $\mathcal{K}\otimes\mathcal{T}\otimes A\hookrightarrow\overline{\mathcal{T}}\otimes A$, the induced map $i_*:K_0(\mathcal{K}\otimes\mathcal{T}\otimes A)\to K_0(\overline{\mathcal{T}}\otimes A)$ is injective too. Denote by $m:\mathcal{T}\otimes A\to\mathcal{K}\otimes\mathcal{T}\otimes A$ the homomorphism given by $m(x\otimes a)=p_0\otimes x\otimes a$, which induces an isomorphism m_* on K-theory by corollary 2.5.3. Then we have

$$i_*\circ m_*\circ(\varepsilon\otimes id_A)_*=(\alpha_1\otimes id_A)_*=(\alpha_0\otimes id_A)_*=i_*\circ m_*\circ(id_{\mathcal{T}\otimes A})_*$$

So, since i_* is injective m_* is an isomorphism, we get that $\ker(\varepsilon\otimes id_A)_*=0$. Thus, by lemma 3.3.1, $\mathbb{K}_0(\mathcal{T}_0\otimes A)=0$. □

3.4 Bott periodicity and 6-term exact sequence

By all the results of the previous subsections, we can easily conclude and finish the proof of the Bott periodicity theorem and then deduce the 6-term exact sequence as follows.

Theorem 3.4.1. Let A be a C^*-algebra. Then $K_{-2}(A)\cong K_0(A)$.

Proof. From the previous subsection, we have the short exact sequence

$$0\to\mathcal{K}\otimes A\to\mathcal{T}_0\otimes A\to\mathcal{S}A\to 0$$

Remember that $\mathcal{S}A\cong C_0(\mathbb{R})\otimes A$ and so $\mathcal{S}(\mathcal{T}_0\otimes A)\cong\mathcal{T}_0\otimes\mathcal{S}A$. Hence the half-infinite exact sequence induced by the short exact sequence above gives us the exact sequence

$$0=K_0(\mathcal{T}_0\otimes\mathcal{S}A)\cong K_{-1}(C_0(\mathbb{R})\otimes A)\xrightarrow{\partial}K_0(\mathcal{K}\otimes A)\to K_0(\mathcal{T}_0\otimes A)=0$$

using the K-contractibility of $\mathcal{T}_0$. Then $\partial:K_{-2}(A)=K_{-1}(C_0(\mathbb{R})\otimes A)\to K_0(\mathcal{K}\otimes A)\cong K_0(A)$ is an isomorphism. □

Proposition 3.4.1. Given a short exact sequence of C^*-algebras

$$0\to I\to A\to A/I\to 0$$

we have the 6-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

Proof. This is a straightforward consequence of Bott periodicity. Indeed, applying the Bott periodicity theorem, we can shorten to the left the half-infinite exact sequence induced by the short exact sequence above as follows :

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d] \\
K_{-1}(A/I) \arrow[u] & K_{-1}(A) \arrow[l] & K_{-1}(I) \arrow[l]
\end{tikzcd}$$

Then we conclude with the fact that $K_{-1} \cong K_1$. $\square$

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